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PRIME IDEALS IN CROSSED PRODUCTS OF FINITE GROUPS

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ABSTRACT
Let $R * G$ be a crossed product of the finite group G over the ring R. In this paper we discuss the relationship between the prime ideals of $R * G$ and the G-prime ideals of R. In particular, we show that Incomparability and Going Down hold in this situation. In the course of the proof, we actually completely describe all the prime ideals P of $R * G$ such that $P \cap R$ is a fixed G-prime ideal of R. As an application, we prove that if G is a finite group of automorphisms of R, then the prime (primitive) ranks of R and of the fixed ring R^G are equal provided $|G|^{-1} \in R$. In an appendix, we extend some of these results to crossed products of the infinite cyclic group.

§1. Introduction

Let G be a multiplicative group and let R be a ring with 1. Then a crossed product $R * G$ of G over R is an associative ring containing for each $x \in G$ an element $\bar{x} \in R * G$. The set $\{\bar{x} \mid x \in G\}$ is a basis of $R * G$ as a left R-module and hence every element $\alpha \in R * G$ can be uniquely written as a finite sum

$$\alpha = \sum_{x \in G} r_x \bar{x}$$

with $r_x \in R$. The addition in $R * G$ is the obvious one and the multiplication is given by the formulas

$$\bar{x}\bar{y} = t(x, y)\overline{xy},$$

$$r\bar{x} = \bar{x}r^{\bar{x}}$$

for all $x, y \in G$ and $r \in R$. Here $t : G \times G \to U$ is a map from $G \times G$ to the

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group of units U of R and, for fixed $x \in G$, the map ${}^{\bar{x}}: r \rightarrow r^{\bar{x}}$ is an automorphism of R.

It is an easy exercise to determine the relations on t and the automorphisms ${}^{\bar{x}}$ which make $R * G$ associative. The ring $R * G$ has an identity element namely $1 = t(1,1)^{-1}\bar{1}$ and hence without loss of generality we will assume throughout that $\bar{1} = 1$. Moreover each $\bar{x}$ is invertible, and in fact

$$\mathfrak{G} = \{u\bar{x} \mid u \in U, x \in G\}$$

is a multiplicative group of units in $R * G$ which acts on R by conjugation when we view R as a subring of $R * G$ via the embedding $r \rightarrow r1$. Thus U is a normal subgroup of $\mathfrak{G}$ with $\mathfrak{G}/U \simeq G$. Observe that conjugation by U stabilizes the ideals of R and thus there is a well defined action of $G \simeq \mathfrak{G}/U$ on the set of these ideals. We say that the G-invariant ideal A of R is G-prime if and only if $B_1 \cdot B_2 \subset A$ for G-invariant ideals B_i of R implies that $B_1 \subset A$ or $B_2 \subset A$.

In this paper we investigate the relationship between prime ideals of $R * G$ and G-prime ideals of R. For the most part, we will assume that G is finite. Indeed only in Section 5, an appendix to this paper, will G be allowed to be infinite. A simple relationship between the above two classes of ideals is as follows.

LEMMA 1.1. *If P is a prime ideal of $R * G$, then $P \cap R$ is a G-prime ideal of R. Conversely, if A is a G-prime ideal of R, then there exists at least one prime ideal P of $R * G$ such that $P \cap R = A$.*

PROOF. Observe that if I is an ideal of $R * G$, then since I is G-invariant so is $I \cap R$. Conversely, if A is a G-invariant ideal of R, then we have easily $A \cdot (R * G) = (R * G) \cdot A$ so $A * G = A \cdot (R * G)$ is a two sided ideal of $R * G$ with $(A * G) \cap R = A$.

If P is a prime ideal of $R * G$, then $P \cap R$ is at least G-invariant. Suppose A and B are G-invariant ideals of R with $AB \subset P \cap R$. Then

$$(A * G)(B * G) = (R * G)AB(R * G) \subset (R * G)(P \cap R)(R * G) \subset P$$

so the primeness of P yields $A * G \subset P$ or $B * G \subset P$. Hence $A \subset P \cap R$ or $B \subset P \cap R$ and $P \cap R$ is G-prime.

Conversely let A be a G-prime ideal of R and observe that $A * G$ is an ideal of $R * G$ with $(A * G) \cap R = A$. Hence by Zorn's lemma we can choose P an ideal of $R * G$ maximal with respect to $P \cap R = A$. If I, J are ideals of $R * G$ properly larger than P, then $I \cap R$ and $J \cap R$ are G-invariant ideals of R

properly larger than A. Since A is G-prime this yields $(I \cap R)(J \cap R) \not\subseteq A$ and hence $IJ \not\subseteq P$. Thus P is prime.

The main questions we are interested in are the following:

Incomparability. If $P_1 \subsetneqq P_2$ are prime ideals of $R * G$, does it follow that $P_1 \cap R \subsetneqq P_2 \cap R$?

Going Down. Given G-primes $A_1 \subset A_2$ of R and a prime P_2 of $R * G$ with $P_2 \cap R = A_2$. Does there exist a prime P_1 of $R * G$ satisfying $P_1 \subset P_2$ and $P_1 \cap R = A_1$?

We remark that the answer to both of these questions is well-known to be positive in case R is (right or left) noetherian. The main task of this paper is to show that this is still true for any ring R. In a later paper, we will settle the Going Up problem. It is easy to see that for Incomparability the primeness of the larger ideal P_2 is unnecessary. Indeed we shall prove in Section 3 that

THEOREM 1.2. *Let $P \subset I$ be ideals of $R * G$ with G finite. If P is prime and $P \neq I$, then $P \cap R \neq I \cap R$.*

Of course in dealing with this we may factor out the ideal $(P \cap R) * G$ of $R * G$ and thus reduce to the case $P \cap R = 0$ which makes R a G-prime ring. In the same way, the Going Down problem can also be reduced to the case of a G-prime coefficient ring R, since A_1 is G-prime and $(R * G)/(A_1 * G) \simeq (R/A_1) * G$. For such rings we have the following result, which is also proved in Section 3.

THEOREM 1.3. *Let $R * G$ be given with G finite and with R a G-prime ring.*

(i) *A prime ideal P of $R * G$ is minimal if and only if $P \cap R = 0$.*

(ii) *There are finitely many such minimal primes, say $P_1, P_2, \cdots, P_n$, and in fact $n \leqq |G|$.*

(iii) *$J = P_1 \cap P_2 \cap \cdots \cap P_n$ is the unique largest nilpotent ideal of $R * G$ and $J^{|G|} = 0$.*

Observe that if P is any prime ideal of $R * G$, as above, then P surely contains the nilpotent ideal $P_1 \cap P_2 \cap \cdots \cap P_n$ and hence $P \supset P_i$ for some i. Thus P contains a prime ideal P_i with $P_i \cap R = 0$ and this is precisely the solution to the Going Down problem.

In proving these results we require two different types of reductions. Section 2 deals with the case when R is prime. Here we set up a one-to-one correspondence between the prime ideals of $R * G$ satisfying $P \cap R = 0$ and the G-prime ideals of a certain finite dimensional algebra E. E is in fact isomorphic to a twisted group algebra $C^t[G_{\mathrm{inn}}]$ where the field C is the extended centroid of R and G_{inn} is a particular normal subgroup of G. In Section 3 we reduce the G-prime case to the case of R being prime by passing from $R * G$ to $R * H$ where H is the subgroup of G stabilizing a minimal prime Q of R.

Section 4 is devoted to some applications of this material to rings with finite group actions. If G is a finite group acting as automorphisms on a ring R, then one can form the skew group ring RG. This is just the special case of a crossed product $R * G$ where $t(x, y) = 1$ for all $x, y \in G$ and where the automorphisms $^{\bar{x}}$ are given by the action of G on R. We use RG to study the relationship between R and the subring R^G consisting of the elements of R that are fixed under G. The above results on crossed products then apply to yield

THEOREM 1.4. *Assume $|G|^{-1} \in R$. Then the prime rank of R is equal to the prime rank of R^G and the primitive rank of R is equal to the primitive rank of R^G.*

Recall that for any ring R, the prime rank is the largest integer n such that R has a chain of prime ideals $P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n$. If no such maximum exists, then, of course, R has infinite prime rank. The primitive rank is defined analogously using primitive ideals instead of primes. Theorem 1.4 generalizes a result of the first named author [6, theorem 2.7] and a result of Fisher and Osterburg [3, proposition 3.4].

In an appendix to this paper (Section 5) we extend some of these results to infinite groups. In particular we show that an analog of Theorem 1.2 holds for groups which have an infinite cyclic subgroup of finite index.

We close this section with a lemma which extends [9, lemma 4]. If T is a subset of G, we let $R * T$ denote the set of elements of $R * G$ with support in T.

LEMMA 1.5. *Let I be a nonzero left and right R-submodule of $R * G$ and let $T = \{x_1, x_2, \cdots, x_n\}$ be a subset of G with $x_1 = 1$. We suppose that $I \cap R * T \neq 0$ and $I \cap R * T' = 0$ for all $T' \subsetneqq T$. For each $i = 1, 2, \cdots, n$ let B_i be defined by*

$$B_i = \{r \in R \mid \text{there exists } \beta = \sum_1^n r_j \bar{x}_j \in I \text{ with } r = r_i\}.$$

Then

(i) *Each B_i is a nonzero ideal of R.*

(ii) *If $A = B_1$, then for each i there exists a natural bijection $f_i : A \to B_i$ which is additive and satisfies $(rat)f_i = r(af_i)t^{\bar{x}_i^{-1}}$ for all $r, t \in R$ and $a \in A$. Here f_1 is the identity map.*

(iii) *The elements of $I \cap R * T$ are precisely the elements of the form*

$$\alpha = \sum_1^n (af_i)\bar{x}_i$$

with $a \in A$.

PROOF. Since I is a left and right R-submodule of $R * G$, it is clear that each B_i is a two sided ideal of R. Furthermore, the minimality of T implies that each B_i is nonzero. Observe also that for each $b_i \in B_i$, the minimality of T implies that there exists a unique $\beta = \Sigma_1^n r_j\bar{x}_j \in I$ with $r_i = b_i$. Indeed if β and β' are two such elements, then $\beta - \beta' \in I$ is an element of smaller support and hence $\beta - \beta' = 0$.

Let $A = B_1$. Then for each $a \in A$ there exists a unique $\alpha = \Sigma_1^n b_i\bar{x}_i \in I$ with $b_1 = a$. We can then define $f_i : A \to B_i$ by setting $af_i = b_i$ and we see immediately that

$$\alpha = \sum_1^n (af_i)\bar{x}_i.$$

Conversely it is clear that every element of $I \cap R * T$ is of this form. Furthermore, each f_i is clearly an additive bijection. Finally let $a \in A$ and α be as above and let $r, t \in R$. Then

$$r\alpha t = \sum_1^n rb_it^{\bar{x}_i^{-1}}\bar{x}_i \in I$$

and since $\bar{x}_1 = 1$, this implies that

$$(rat)f_i = rb_it^{\bar{x}_i^{-1}} = r(af_i)t^{\bar{x}_i^{-1}}.$$

The lemma is proved.

§2. Prime coefficient rings

In this section G will always denote a finite group and R will be a prime ring (with 1).

We begin by briefly discussing a certain ring of quotients $S = Q_0(R)$ which is defined in [7] essentially as follows. Consider the set of all left R-module homomorphisms $f : {}_RA \to {}_RR$ where A ranges over all nonzero two sided ideals of R. Two such functions are said to be equivalent if they agree on their common domain, which is a nonzero ideal since R is prime. It is easy to check that this defines an equivalence relation (see [7] or [8]). We let $\hat{f}$ denote the equivalence class of f and we let $S = Q_0(R)$ be the set of all such equivalence classes.

The arithmetic in S is defined in the following manner. Suppose $f : {}_RA \to {}_RR$ and $g : {}_RB \to {}_RR$. Then $\hat{f} + \hat{g}$ is the class of $f + g : {}_R(A \cap B) \to {}_RR$ and $\hat{f}\hat{g}$ is the class of the composite function $fg : {}_R(BA) \to {}_RR$. It is easy to see that these definitions make sense and that they respect the equivalence relation. Furthermore the ring axioms are surely satisfied so S is in fact a ring with 1. Finally let $a_\rho : {}_RR \to {}_RR$ denote right multiplication by $a \in R$. Then the map $a \to \hat{a}_\rho$ is easily seen to be a ring homomorphism from R into S. Moreover, if $a \neq 0$, then $Ra_\rho \neq 0$ and hence $\hat{a}_\rho \neq 0$. We conclude therefore that R is embedded isomorphically in S and hence we will view R as a subring of S with the same 1.

The first two parts of the following lemma are contained in [7] as well as in [8]. Part (iii) is due to Kharchenko in [5]. We include them for the sake of completeness.

LEMMA 2.1. *Let $S = Q_0(R)$ be as above.*

(i) *If $s \in S$ and $As = 0$ for some nonzero ideal A of R, then $s = 0$.*

(ii) *If $s_1, s_2, \cdots, s_n \in S$, then there exists a nonzero ideal A of R with $As_1, As_2, \cdots, As_n \subset R$.*

(iii) *Let σ be an automorphism of R and let A and B be nonzero ideals of R. Suppose that $f : A \to B$ is an additive bijection which satisfies*

$$(rat)f = r(af)t^\sigma$$

for all $r, t \in R$ and $a \in A$. Then $s = \hat{f}$ is a unit in S, conjugation by s induces the automorphism σ on R and $af = as$ for all $a \in A$.

PROOF. Suppose $g : {}_RB \to {}_RR$ and $b \in B$. Then $b_\rho g$ is defined on ${}_RR$ and for all $r \in R$ we have

$$r(b_\rho g) = (rb)g = r(bg) = r(bg)_\rho.$$

Hence $\hat{b}_\rho \hat{g} = \widehat{(bg)}_\rho$ and the map g translates in S to right multiplication by $\hat{g}$.

(i) Let $s \in S$ with $As = 0$. If $s = \hat{g}$, then the above shows that g vanishes on an ideal in its domain and hence $s = \hat{g} = 0$.

(ii) Let $s_1, s_2, \cdots, s_n \in S$ with $s_i = \hat{g}_i$. Then we can surely assume that all g_i are defined on the common domain A, since R is prime. From this we have $As_i = Ag_i \subset R$ for all i.

(iii) Since f is a bijection, there exists a back map $g : B \to A$ which is also an additive bijection. Furthermore $(rbt)g = r(bg)t^{\sigma^{-1}}$ for all $r, t \in R$ and $b \in B$. Observe that f and g are left R-module homomorphisms so $s = \hat{f}$ and $\hat{g}$ are elements of S. Moreover, since fg is the identity map on A and gf is the identity on B, we conclude that $\hat{g} = s^{-1}$.

Let $r \in R$. Then $gr_\rho f$ is defined on B and for all $b \in B$ we have

$$b(gr_\rho f) = (bg)r \cdot f = (br^\sigma)gf = br^\sigma = br_\rho^\sigma.$$

Hence $gr_\rho f = r_\rho^\sigma$ and this translates in S to yield $s^{-1}rs = r^\sigma$ for all $r \in R$. Finally since $s = \hat{f}$ we have $af = as$ for all $a \in A$.

It can be shown that any automorphism of R extends uniquely to an automorphism of S. In particular the automorphisms $^{\bar{x}}$ of R given by the crossed product $R * G$ can be so extended and we denote the resulting automorphisms of S by the same symbol. This observation allows us to extend $R * G$ to a crossed product $S * G$. Details can be found in [8, lemmas 2.1 and 2.3]. For the remainder of this section the following notation will be fixed.

NOTATION. $R * G$ will be viewed as a subring of $S * G$. The center of S will be denoted by C and E will denote the centralizer of S in $S * G$. C is usually called the extended centroid of R.

An automorphism σ of R is said to be X-inner if and only if it is induced by conjugation by a unit of $S = Q_0(R)$. In other words, these automorphisms arise from those units $s \in S$ with $s^{-1}Rs = R$. It is easily seen that the set of all X-inner automorphisms of R is in fact a normal subgroup of the group of all automorphisms of R (see [5]). Recall that $\mathfrak{G} = \{u\bar{x} \mid u \in U, x \in G\}$ acts on R and the elements of U surely act as X-inner automorphisms. Thus since $\mathfrak{G}/U \simeq G$ we see that

$$G_{\text{inn}} = \{x \in G \mid {}^{\bar{x}} \text{ is an } X\text{-inner automorphism of } R\}$$

is a normal subgroup of G. The group $\mathfrak{G}$ also acts on S and on $S * G$ and hence acts on E. But U clearly acts trivially on E, since E centralizes R, and so in fact G acts on E. The next lemma contains the necessary information about C and E.

LEMMA 2.2. *Let C and E be as above. Then*

(i) *C is a field.*

(ii) *E is a finite dimensional C-algebra. In fact $E = C^t[G_{\text{inn}}]$, some twisted group algebra of G_{inn} over C.*

(iii) $S * G_{\text{inn}} = S \otimes_C E$.

(iv) *If L is a G-invariant ideal of E, then $L(S * G) = (S * G)L$ is an ideal of $S * G$. Moreover, $(S * G)L$ considered as a left S-submodule of $S * G$ is a direct summand of $S * G$. Furthermore, $(S * G)L \cap (S * G_{\text{inn}}) = SL$ and if $L \neq E$ then $(S * G)L \cap S = 0$.*

Proof. Assertions (i), (ii), (iii) are proved in [8, lemmas 2.1 and 2.3]. For part (iv) let L be a G-invariant ideal of E. Then by definition of E we have $LS = SL$ and the G-invariance gives $L\bar{x} = \bar{x}L$ for all $x \in G$. Thus we see that $L(S * G) = (S * G)L$ is an ideal of $S * G$.

Let Y be a transversal for G_{inn} in G and let L' be a C-complement for L in E. The latter surely exists since C is a field. Then by (iii) we have $S * G_{\text{inn}} = SL \oplus SL'$ and hence clearly

$$S * G = \sum_{y \in Y} SL\bar{y} \oplus \sum_{y \in Y} SL'\bar{y}.$$

Moreover since L is G-invariant

$$(S * G)L = \sum_{y \in Y} (S * G_{\text{inn}})L\bar{y} = \sum_{y \in Y} SL\bar{y}$$

and we conclude that $(S * G)L$ is an S-direct summand of $S * G$. Furthermore, $\Sigma_{y \in Y} SL\bar{y}$ is surely a direct sum and hence we have $(S * G)L \cap (S * G_{\text{inn}}) = SL$. Finally, if $L \neq E$, then we can choose L' above to contain the identity element 1. This then implies that $\Sigma_{y \in Y} SL'\bar{y}$ contains S and hence $(S * G)L \cap S = 0$.

As we have already remarked in the introduction, we handle the case of prime coefficient rings by setting up a one-to-one correspondence between the prime ideals of $R * G$ satisfying $P \cap R = 0$ and the G-prime ideals of E. The following definition describes this association more generally.

Definition. (i) If L is a G-invariant ideal of E, then we set

$$L^u = L(S * G) \cap R * G$$

so that L^u is an ideal of $R * G$, by Lemma 2.2 (iv).

(ii) For any ideal I of $R * G$ we set

$$I^d = \{\gamma \in E \mid A\gamma \subset I \text{ for some nonzero ideal } A \text{ of } R\}.$$

I^d is easily seen to be a G-invariant ideal of E. Indeed suppose $\gamma_1, \gamma_2 \in I^d$

with $A_1\gamma_1, A_2\gamma_2 \subset I$ and let $\delta \in E$. By Lemma 2.1 (ii) there exists a nonzero ideal B of R with $B\delta \subset R * G$. Hence since γ_1, γ_2 and δ commute with R we have

$$(A_1 \cap A_2)(\gamma_1 + \gamma_2) \subset I,$$

$$A_1 B(\gamma_1\delta) = (A_1\gamma_1)(B\delta) \subset I$$

and

$$BA_1(\delta\gamma_1) = (B\delta)(A_1\gamma_1) \subset I.$$

Thus $\gamma_1 + \gamma_2, \gamma_1\delta, \delta\gamma_1 \in I^d$ so I^d is an ideal of E which is clearly G-invariant.

We point out that the definition of the ideal I^d is already implicit in work of Fisher and Montgomery [2]. Clearly the maps ^{u} and ^{d} are monotone. We next observe two elementary facts concerning the ideals L^u. For this we need the following

DEFINITION. Let I be an ideal of $R * G$. We say that I is R-cancelable if and only if for any $\alpha \in R * G$ and any nonzero G-invariant ideal A of R, $A\alpha \subset I$ implies that $\alpha \in I$.

It is clear that any prime ideal of $R * G$ satisfying $P \cap R = 0$ is R-cancelable. Part (i) of the following lemma offers other examples of R-cancelable ideals that will become important later on.

LEMMA 2.3. *Let L, L_1 and L_2 be G-invariant ideals of E. Then*
(i) *L^u is R-cancelable.*
(ii) $L_1^u \cdot L_2^u \subset (L_1L_2)^u$.

PROOF. (i) Suppose $A\alpha \subset L^u$, where $\alpha \in R * G$ and A is a nonzero G-invariant ideal of R. Then, by definition of L^u, it follows that $A\alpha \subset (S * G)L$. Now by Lemma 2.2 (iv) we know that $(S * G)L$ is a left S-module direct summand of $S * G$ so $S * G = (S * G)L \oplus K$ for some S-submodule K of $S * G$. Writing $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \in (S * G)L$, $\alpha_2 \in K$ we see that

$$A(\alpha - \alpha_1) = A\alpha_2 \in (S * G)L \cap K = 0.$$

Hence Lemma 2.1 (i) implies that $\alpha = \alpha_1$ and so $\alpha \in (S * G)L \cap R * G = L^u$. Thus L^u is in fact R-cancelable.

(ii) By Lemma 2.2 (iv) we have $(S * G)L_i = L_i(S * G)$ and thus

$L_1^u \cdot L_2^u \subset L_1(S*G)L_2(S*G) = L_1L_2(S*G)$. Moreover $L_1^u \cdot L_2^u \subset R*G$ so $L_1^u \cdot L_2^u \subset L_1L_2(S*G) \cap R*G = (L_1L_2)^u$.

The next lemma is the crux of our argument.

Lemma 2.4. *With the above notation we have*
(i) *For any G-invariant ideal L of E we have $L = L^{ud}$.*
(ii) *If I is an ideal of $R*G$, then $I \subset I^{du}$.*

Proof. (i) If $\gamma \in L$, then by Lemma 2.1 (ii) there exists a nonzero ideal B of R with $B\gamma \subset R*G$. Then $B\gamma \subset L(S*G) \cap R*G = L^u$ and so $\gamma \in L^{ud}$. Thus $L \subset L^{ud}$. Conversely, if $\gamma \in L^{ud}$, then $A\gamma \subset L^u$ for some nonzero ideal A of R. Thus

$$A\gamma \subset (S*G)L \cap (S*G_{\text{inn}}) = SL,$$

by Lemma 2.2 (iv). Let L' be a C-complement for L in E and write $\gamma = \gamma_1 + \gamma_2$ with $\gamma_1 \in L$ and $\gamma_2 \in L'$. Then $A(\gamma - \gamma_1) = A\gamma_2 \in SL \cap SL' = 0$ and Lemma 2.1 (i) implies that $\gamma = \gamma_1 \in L$.

(ii) Let $\alpha \in I$. We show that $\alpha \in I^{du}$ by induction on $|\operatorname{Supp}\alpha|$, the case $|\operatorname{Supp}\alpha| = 0$ being trivial. Thus suppose α is nonzero and that the result is known for all elements $\gamma \in I$ of smaller support size. Choose $T \subset \operatorname{Supp}\alpha$ minimal with respect to the property that $I \cap R*T \neq 0$. If $y \in T$, then $\operatorname{Supp}\alpha\bar{y}^{-1} = (\operatorname{Supp}\alpha)y^{-1} \supset Ty^{-1}$, Ty^{-1} also has this minimal property and $1 \in Ty^{-1}$. Since it clearly suffices to show that $\alpha\bar{y}^{-1} \in I^{du}$, we can replace α by $\alpha\bar{y}^{-1}$ and T by Ty^{-1} and hence we can assume that $1 \in T$.

If $T = \{x_1 = 1, x_2, \cdots, x_n\}$, then Lemma 1.5 applies and we use its notation. In particular there exist nonzero ideals $A = B_1, B_2, \cdots, B_n$ of R and additive bijections $f_i : A \to B_i$ satisfying

$$(rat)f_i = r(af_i)t^{\bar{x}_i^{-1}}$$

for all $a \in A$ and $r, t \in R$. Thus by Lemma 2.1 (iii) there exist units $s_1, s_2, \cdots, s_n \in S$ such that conjugation by s_i induces the automorphism $\bar{x}_i^{-1}$ on R and $af_i = as_i$ for all $a \in A$. Moreover since f_1 is the identity function we have $s_1 = \hat{f}_1 = 1$.

Let $\beta = \sum s_i\bar{x}_i \in S*G$. Since conjugation by s_i induces the automorphism $\bar{x}_i^{-1}$ on R we see that $s_i\bar{x}_i$ centralizes R. But $s_i\bar{x}_i$ then yields an automorphism of S acting trivially on R and, by the uniqueness of extension of automorphisms from R to S, we conclude that $s_i\bar{x}_i$ must centralize S. Thus $s_i\bar{x}_i \in E$, the centralizer of S

in $S*G$, and $\beta \in E$. Furthermore, since $af_i = as_i$ for $a \in A$, it follows from Lemma 1.5 (iii) that for all $a \in A$

$$a\beta = \Sigma\, as_i\bar{x}_i = \Sigma\, af_i\bar{x}_i \in I.$$

Hence, by definition, we have $\beta \in I^d$.

Let $r = tr\,\alpha$ be the identity coefficient of α and let $a \in A$. Define

$$\gamma = a\alpha - a\beta r.$$

Since $\operatorname{Supp}\beta = T \subset \operatorname{Supp}\alpha$, $1 \in T$ and $s_1 = 1$, we see that $|\operatorname{Supp}\gamma| < |\operatorname{Supp}\alpha|$. Moreover, $\gamma \in I$ since $a\beta \in I$, so by induction we have $\gamma \in I^{du}$. Now $\beta \in I^d$ and $a\beta r \in R*G$ so $a\beta r \in I^{du}$ and hence we conclude that $a\alpha \in I^{du}$ for all $a \in A$. Thus if $D = \bigcap_{x \in G} A^{\bar{x}}$, then D is a nonzero G-invariant ideal of R with $D\alpha \subset I^{du}$. Since I^{du} is R-cancelable, by Lemma 2.3 (i), we deduce from this that $\alpha \in I^{du}$. The result follows by induction.

We now come to the main theorem of this section. We will prove it simultaneously with Lemma 2.6 which is of course a special case of Theorem 1.2.

THEOREM 2.5. *Let $R*G$ be a crossed product of the finite group G over the prime ring R and let $E = C^t[G_{\text{inn}}]$ be the centralizer of $S = Q_0(R)$ in $S*G$. Then the maps ^{d} and ^{u} yield a one-to-one correspondence between the prime ideals P of $R*G$ with $P \cap R = 0$ and the G-prime ideals of E. More precisely:*

(i) *If P is a prime ideal of $R*G$ with $P \cap R = 0$, then P^d is a G-prime ideal of E and $P = P^{du}$.*

(ii) *If L is a G-prime ideal of E, then L^u is a prime ideal of $R*G$ with $L^u \cap R = 0$ and $L = L^{ud}$.*

LEMMA 2.6. *Let $R*G$ be given with R prime. If P is a prime ideal of $R*G$ with $P \cap R = 0$ and if I is an ideal of $R*G$ properly containing P, then $I \cap R \neq 0$.*

PROOF. (i) Let P be a prime ideal of $R*G$ with $P \cap R = 0$. We first show that $P = P^{du}$. By Lemma 2.4 (ii) we have at least $P \subset P^{du}$. Now let $\alpha \in P^{du}$. Then $\alpha \in R*G$ and $\alpha \in (S*G)P^d$ so $\alpha = \Sigma_1^n \beta_i\delta_i$ with $\beta_i \in S*G$ and $\delta_i \in P^d$. By definition of P^d there exist nonzero ideals D_i of R with $D_i\delta_i = \delta_i D_i \subset P$. Thus setting $D = \bigcap_{x\in G}\bigcap_{i=1}^n D_i^{\bar{x}}$, we see that D is a nonzero G-invariant ideal of the prime ring R with $\delta_i D \subset P$ for all i. Furthermore, by Lemma 2.1 (ii) there exists a nonzero ideal B of R such that $B\beta_i \subset R*G$ for all i and by the above argument we can also assume that B is G-invariant. Thus

$$\begin{aligned}(B*G)\alpha(D*G) &= (R*G)B\alpha D(R*G)\\ &\subset \sum_i (R*G)B\beta_i\cdot\delta_i D(R*G)\\ &\subset P.\end{aligned}$$

Now P is prime and P does not contain either $B*G$ or $D*G$ since $P\cap R=0$. Hence we deduce that $\alpha\in P$. Since α was arbitrary we obtain the reverse inclusion $P\supset P^{du}$ and have shown that $P=P^{du}$.

Now suppose that M_1 and M_2 are G-invariant ideals of E with $M_1M_2\subset P^d$. Then by Lemma 2.3 (ii), $M_1^uM_2^u\subset(M_1M_2)^u\subset P^{du}=P$. Since P is prime this yields $M_i^u\subset P$ for some i and Lemma 2.4 (i) implies that $M_i=M_i^{ud}\subset P^d$. Thus P^d is G-prime.

(ii) Let L be a G-prime ideal of E. Then since E is a finite dimensional algebra over the field C we see that L is in fact G-maximal. Moreover it follows immediately from Lemma 2.2 (iv) that $L^u\cap R=0$ and by Lemma 2.4 (i) we have $L=L^{ud}$.

Now suppose I is any ideal of $R*G$ properly containing L^u. Then $I^d\supset L^{ud}=L$. But $I^d\neq L$ since otherwise we would have, by Lemma 2.4 (ii), $I\subset I^{du}=L^u$. Thus the G-maximality of L implies that $I^d=E$ and hence $I\cap R\neq 0$.

We conclude from this that L^u is prime. Indeed if $I_1,I_2\supsetneqq L^u$, then $I_1\cap R\neq 0$ and $I_2\cap R\neq 0$ so $0\neq(I_1\cap R)(I_2\cap R)\subset I_1I_2\cap R$. Since $L^u\cap R=0$, this yields $I_1I_2\not\subset L^u$ and Theorem 2.5 is proved.

Finally we prove Lemma 2.6. By (i) above any prime P with $P\cap R=0$ is of the form L^u for the G-prime ideal $L=P^d$. Hence as we have just shown, if $I\supsetneqq P=L^u$ then $I\cap R\neq 0$.

As a corollary we now offer the special case of Theorem 1.3 with R prime.

LEMMA 2.7. *Let $R*G$ be given with G finite and with R prime.*

(i) *A prime ideal P of $R*G$ is minimal if and only if $P\cap R=0$.*

(ii) *There are finitely many such minimal primes, say $P_1,P_2,\cdots,P_n$ and in fact $n\leqq|G_{\rm inn}|\leqq|G|$.*

(iii) *$J=P_1\cap P_2\cap\cdots\cap P_n$ is the unique largest nilpotent ideal of $R*G$. In fact $J=(\operatorname{rad}E)^u$, where $\operatorname{rad}E$ is the Jacobson radical of E, and $J^{|G_{\rm inn}|}=0$.*

PROOF. Let $L_1,L_2,\cdots,L_n$ be all the G-prime ideals of E. Then clearly

$$n\leqq\dim_C C^t[G_{\rm inn}]=|G_{\rm inn}|\leqq|G|$$

and also $L_1 \cap L_2 \cap \cdots \cap L_n = \operatorname{rad} E$, the Jacobson radical of E. If $P_i = L_i^u$, then by Theorem 2.5 we know that $P_1, P_2, \cdots, P_n$ are the unique prime ideals of $R * G$ which are disjoint from $R\backslash\{0\}$.

Set $J = P_1 \cap P_2 \cap \cdots \cap P_n$. Then $J \subset L_i^u$ so $J^d \subset L_i^{ud} = L_i$, by Theorem 2.5, and hence

$$J^d \subset L_1 \cap L_2 \cap \cdots \cap L_n = \operatorname{rad} E.$$

Lemma 2.4 (ii) now yields $J \subset J^{du} \subset (\operatorname{rad} E)^u$ and we conclude immediately from Lemma 2.3 (ii) and the nilpotence of $\operatorname{rad} E$ that $(\operatorname{rad} E)^u$ and J are nilpotent. On the other hand, each P_i certainly contains all nilpotent ideals of $R * G$, so $J = P_1 \cap P_2 \cap \cdots \cap P_n$ contains $(\operatorname{rad} E)^u$ and $J = (\operatorname{rad} E)^u$ is clearly the largest nilpotent ideal of $R * G$. Furthermore, since $(\operatorname{rad} E)^{|G_{\text{inn}}|} = 0$, we have $J^{|G_{\text{inn}}|} = 0$.

Finally, if P is any prime ideal of $R * G$ then, since J is nilpotent, $P \supset J = P_1 \cap P_2 \cap \cdots \cap P_n$ and hence $P \supset P_i$ for some i. Thus the minimal primes of $R * G$ are the minimal members of the set $\{P_1, P_2, \cdots, P_n\}$. But observe that $P_i \supset P_j$ implies that $L_i = P_i^d \supset P_j^d = L_j$ and hence, since L_j is G-maximal, we must have $i = j$. This shows that $P_1, P_2, \cdots, P_n$ are precisely the minimal primes of $R * G$ and the result follows.

We close this section with a technical lemma which will be needed to prove Theorem 1.3.

LEMMA 2.8. *Let $J = P_1 \cap P_2 \cap \cdots \cap P_n$ be as in the preceding lemma. Then*

(i) *There exists a sequence $J = J_1, J_2, \cdots$ of R-cancelable ideals in $R * G$ such that $JJ_i \subset J_{i+1}$ and $J_k = 0$ for some $k \leqq |G_{\text{inn}}|$.*

(ii) *Let $\alpha_1, \alpha_2, \cdots, \alpha_m$ be elements of J and let A be a nonzero ideal of R. Then there exists a nonzero G-invariant ideal B of R such that $B\alpha_i \subset JA$ for all i.*

PROOF. Set $L = \operatorname{rad} E$ so we know that L is nilpotent and, by Lemma 2.7 (iii), that $J = L^u$.

(i) Set $J_i = (L^i)^u$. Then the ideals J_i are R-cancelable, by Lemma 2.3 (i), and $JJ_i = L^u(L^i)^u \subset (LL^i)^u = (L^{i+1})^u = J_{i+1}$ by Lemma 2.3 (ii). Moreover, since $L^k = 0$ for some $k \leqq |G_{\text{inn}}|$, it follows that $J_k = 0$.

(ii) Let $\gamma = \beta s\bar{g} \in S * G$ with $\beta \in L$, $s \in S$ and $g \in G$. We show first that there exists a nonzero G-invariant ideal D of R such that $D\gamma \subset JA$. By Lemma 2.1 (ii) we can choose nonzero ideals D_1 and D_2 of R with $D_1\beta \subset (R * G) \cap (S * G)L = L^u$ and $D_2 s \subset R$ and by taking the intersections of the G-conjugates of these ideals we may even assume that D_1 and D_2 are G-invariant. By the same device we can find a nonzero G-invariant ideal D_3 of R with $D_3 \subset A$. Now let $D = D_1D_3D_2$. Then since β commutes with R and D_3 is G-invariant we have

$$D\gamma = (D_1\beta)D_3(D_2s)\bar{g} \subset L^u D_3 R\bar{g} = L^u \bar{g} D_3 \subset JA.$$

Finally, if $\alpha_1, \alpha_2, \cdots, \alpha_n$ are elements of $J = L^u$, then each α_i is contained in $L(S * G)$ and hence can be written as a finite sum $\alpha_i = \Sigma_j \gamma_{ij}$ where $\gamma_{ij} = \beta_{ij}s_{ij}\bar{g}_{ij}$ and $\beta_{ij} \in L$, $s_{ij} \in S$, $g_{ij} \in G$. As above, we can find nonzero G-invariant ideals D_{ij} of R with $D_{ij}\gamma_{ij} \subset JA$. Thus the finite intersection $B = \bigcap_{i,j} D_{ij}$ is an ideal which satisfies the requirements of the lemma.

§3. G-prime coefficient rings

This section contains the proofs of Theorems 1.2 and 1.3. Throughout, G will denote a finite group and $R * G$ will be a crossed product of G over R. Then there is a well defined action of G on the ideals of R and we will assume that R is a G-prime ring. Recall that this means, by definition, that the product of any two nonzero G-invariant ideals of R is nonzero. Certainly, this condition is satisfied if there exists a prime Q in R such that $\bigcap_{x \in G} Q^x = 0$. For, if A_1 and A_2 are G-invariant ideals of R with $A_1A_2 = 0$, then $A_1A_2 \subset Q$ and so $A_i \subset Q$ for some i. Using the G-invariance of A_i we deduce that $A_i \subset \bigcap_{x \in G} Q^x$ and hence $A_i = 0$. Part (i) of the following lemma shows that, conversely, in any G-prime ring R one can find such a prime Q.

We remark on a simple property of semiprime rings. Suppose R is semiprime and let A and B be ideals of R with $AB = 0$. Then $(BA)^2 = 0$ so $BA = 0$. In view of this, left and right annihilators of ideals are equal and we will just use the notation "ann".

LEMMA 3.1. *Let $R * G$ be given and assume that R is G-prime. Then*

(i) *R contains a prime ideal Q with $\bigcap_{x \in G} Q^x = 0$. In particular, R is semiprime.*

(ii) *Any prime of R contains a conjugate Q^x of Q and so $\{Q^x \mid x \in G\}$ are precisely the minimal primes of R.*

(iii) *Let H denote the stabilizer of Q in G and let $N = \operatorname{ann} Q$. Then H is a subgroup of G,*

$$N = \bigcap_{x \notin H} Q^x \neq 0$$

and

$$0 = NN^{\bar{x}} = N^{\bar{x}}N = N\bar{x}N$$

for all $x \in G\backslash H$.

PROOF. (i) Since G is finite, an easy application of Zorn's lemma shows that there exists an ideal Q of R maximal with respect to the property that $\bigcap_{x\in G} Q^{\bar{x}} = 0$. Now suppose A_1 and A_2 are ideals of R containing Q with $A_1A_2 \subset Q$ and set $B_i = \bigcap_{x\in G} A_i^{\bar{x}}$. Then B_1 and B_2 are G-invariant and since $B_1B_2 \subset A_1A_2$ we have

$$B_1B_2 \subset \bigcap_{x\in G} (A_1A_2)^{\bar{x}} \subset \bigcap_{x\in G} Q^{\bar{x}} = 0.$$

Since R is G-prime, we conclude that $B_i = 0$ for some i and then the maximality of Q implies that $A_i = Q$. Thus Q is a prime ideal of R.

(ii) Any prime ideal of R certainly contains $\bigcap_{x\in G} Q^{\bar{x}} = 0$ and consequently contains some $Q^{\bar{x}}$, since G is finite. Furthermore, there are no inclusion relations between the primes $\{Q^{\bar{x}} \mid x \in G\}$. For, if $Q \subsetneqq Q^{\bar{x}}$, then $Q \subsetneqq Q^{\bar{x}^n}$ for all $n \geqq 1$, so by taking $n = |G|$ we obtain the contradiction $Q \subsetneqq Q$.

(iii) If N denotes the annihilator of Q, then $NQ = 0$ yields $NQ \subset Q^{\bar{x}}$ for all $x \in G$. Thus if $x \in G\backslash H$ we deduce from (ii) that $N \subset Q^{\bar{x}}$ and we have shown that $N \subset \bigcap_{x\notin H} Q^{\bar{x}}$. Conversely since

$$\Big(\bigcap_{x\notin H} Q^{\bar{x}}\Big) Q \subset \bigcap_{x\in G} Q^{\bar{x}} = 0$$

we have $N = \operatorname{ann} Q \supset \bigcap_{x\notin H} Q^{\bar{x}}$ and therefore equality occurs. We remark that if $H = G$, then by definition $N = \bigcap_{x\notin H} Q^{\bar{x}} = R$. In any case, by (ii) above we have $Q \not\supset N$ and hence $N \neq 0$. Finally suppose $x \notin H$. Then clearly $N \subset Q^{\overline{x^{-1}}} = Q^{\bar{x}^{-1}}$ so $N^{\bar{x}} \subset Q$ and we have $NN^{\bar{x}} \subset NQ = 0$. Similarly $N^{\bar{x}}N = 0$ and also $0 = \bar{x}N^{\bar{x}}N = N\bar{x}N$.

The notation of the preceding lemma will be kept throughout this section.

NOTATION. Q will denote a minimal prime of the G-prime ring R, N will be its annihilator in R and H will denote the stabilizer of Q in G. Moreover, we set $M = \Sigma_{x\in G} N^{\bar{x}}$ so that M is a nonzero G-invariant ideal of R.

LEMMA 3.2. *Let N and Q be as above.*

(i) $N \cap Q = 0$ *and if A is any nonzero ideal of R with $A \subset N$, then* $\operatorname{ann} A = Q$.

(ii) *Suppose* σ *is an automorphism of* R *and* $f: A \to B$ *is an additive bijection satisfying*

$$(rat)f = r(af)t^{\sigma}$$

for all $r, t \in R$ *and* $a \in A$. *If* A, B *are nonzero ideals of* R *contained in* N, *then* $Q^{\sigma} = Q$.

Proof. (i) By Lemma 3.1 (i), (iii) we have $N \cap Q = 0$ and $NQ = 0$. Thus if $A \subset N$ then $Q \subset \operatorname{ann} A$. Conversely, suppose $AB = 0$. If $A \neq 0$ then $A \not\subset Q$ by the above and hence $AB = 0 \subset Q$ implies that $B \subset Q$. Thus $Q = \operatorname{ann} A$.

(ii) Let f and σ be given. Then $AQ = 0$ yields

$$0 = (AQ)f = (Af)Q^{\sigma} = BQ^{\sigma}.$$

Thus $Q^{\sigma} \subset \operatorname{ann} B = Q$. On the other hand, since $BQ = 0$ and f is one-to-one,

$$0 = BQ = (Af)Q = (AQ^{\sigma^{-1}})f$$

yields $AQ^{\sigma^{-1}} = 0$. Thus $Q^{\sigma^{-1}} \subset \operatorname{ann} A = Q$ so $Q \subset Q^{\sigma}$ and we deduce that Q is σ-invariant.

Note that $Q * H$ is a two sided ideal of $R * H$. Roughly speaking, our method in this section is to pass from $R * G$ to $(R * H)/(Q * H) \simeq (R/Q) * H$ and thus reduce the general problem to the case of prime coefficient rings where the results of Section 2 can be applied. The following definition introduces the necessary machinery.

Definition. (i) For any ideal L of $R * H$ we set

$$L^{\vee} = \{\alpha \in R * G \mid M\alpha \subset \bar{G}NL\bar{G}\}.$$

Since M is G-invariant, $L^{\vee}$ is clearly an ideal of $R * G$.

(ii) If I is an ideal of $R * G$, then we set

$$I^{\delta} = \{\alpha \in R * H \mid N\alpha \subset I\}.$$

Since $N = \operatorname{ann} Q$ is H-invariant, I^{δ} is an ideal of $R * H$. Moreover $N(Q * H) = 0 \subset I$ shows that $I^{\delta} \supset Q * H$.

Obviously, the maps $^{\vee}$ and $^{\delta}$ are monotone, as are the maps ^{u} and ^{d} used in Section 2. In fact, as we will see, the maps $^{\vee}$ and $^{\delta}$ behave similarly to ^{u} and ^{d} in many other respects. Indeed the following two lemmas are the analogs of Lemmas 2.3 and 2.4.

LEMMA 3.3. *If L, L_1 and L_2 are ideals of $R*H$, then*
(i) $NL\bar{G}N \subset NL$.
(ii) $L_1^\nu M L_2^\nu \subset (L_1L_2)^\nu$.

PROOF. (i) Observe that $NL \subset (R*H)N$ since N is H-invariant. Therefore, for any $x \in G\backslash H$ we have

$$NL\bar{x}N \subset (R*H)N\bar{x}N = 0,$$

where the latter equality follows from Lemma 3.1 (iii). Consequently, $NL\bar{G}N = NL\bar{H}N \subset NL$.

(ii) Clearly,

$$M(L_1^\nu ML_2^\nu) = (ML_1^\nu)(ML_2^\nu) \subset (\bar{G}NL_1\bar{G})(\bar{G}NL_2\bar{G}) = \bar{G}(NL_1\bar{G}N)L_2\bar{G},$$

by definition of L_i^ν, and hence by part (i), $M(L_1^\nu ML_2^\nu) \subset \bar{G}NL_1L_2\bar{G}$. Thus $L_1^\nu ML_2^\nu \subset (L_1L_2)^\nu$.

LEMMA 3.4. *With the above notation we have*
(i) *Let L be an ideal of $R*H$. Then $L \subset L^{\nu\delta}$.*
(ii) *If I is an ideal of $R*G$, then $MI^{\delta\nu} \subset I$. More importantly, there exists a nonzero G-invariant ideal E of R with*

$$EI \subset \bar{G}NI^\delta\bar{G} \subset I^{\delta\nu}.$$

PROOF. (i) By definition, $L^{\nu\delta} = \{\alpha \in R*H \mid N\alpha \subset L^\nu\}$. Since $NL \subset \bar{G}NL\bar{G} \subset L^\nu$, we see that $L \subset L^{\nu\delta}$.

(ii) Set $L = I^\delta$. Then, by definition of $^\delta$ and $^\nu$, we have $NL \subset I$ and hence $MI^{\delta\nu} = ML^\nu \subset \bar{G}NL\bar{G} \subset I$.

We now wish to obtain a reverse inclusion relating I and $I^{\delta\nu}$. Suppose first that $NI = 0$. Then since I is G-invariant we have

$$MI = (\Sigma N^{\bar{x}})I = 0 \subset I^{\delta\nu}$$

and the result follows. Thus we may assume that $NI \neq 0$. Observe that NI is a right ideal of $R*G$ and a left R-submodule of $R*G$.

Let $\mathcal{T}$ denote the set of all subsets T of G such that $NI \cap R*T \neq 0$, $NI \cap R*T' = 0$ for all $T' \subsetneq T$ and $1 \in T$. Then $\mathcal{T}$ is a finite nonempty set of subsets of G. Indeed if $\gamma \neq 0$ is an element of NI of minimal support size, then $T = (\operatorname{Supp}\gamma)x^{-1}$, for any $x \in \operatorname{Supp}\gamma$, clearly satisfies the above conditions.

Let $T = \{x_1 = 1, x_2, \cdots, x_n\} \in \mathcal{T}$. Then, by definition of $\mathcal{T}$, Lemma 1.5 applies to the R-bimodule NI. In particular in the notation of that lemma there exist nonzero ideals A, B_i for $i = 1, 2, \cdots, n$ and additive bijections $f_i : A \to B_i$ such that

$$(rat)f_i = r(af_i)t^{\bar{x}_i^{-1}}$$

for all $r, t \in R$ and $a \in A$. Furthermore, since $NI \subset N(R * G)$ we have $A, B_i \subset N$. It now follows immediately from Lemma 3.2 (ii) that $Q^{\bar{x}_i^{-1}} = Q$. In other words, $x_i \in H$ for all i and we have shown that $T \subset H$ for all such $T \in \mathcal{T}$.

For each $T \in \mathcal{T}$, let A_T denote the ideal A, depending upon T, as given above and set $D = \bigcap_{T \in \mathcal{T}} A_T$. Since $\mathcal{T}$ is finite and $A_T \subset N$, it follows from Lemma 3.2 (i) that $D \neq 0$. We show now, by induction on $m = |\operatorname{Supp} \alpha|$, that if $\alpha \in NI$ then $D^{m+1}\alpha \subset \bar{G}NI^\delta\bar{G}$. The case $m = 0$ is of course trivial.

Now let $\alpha \in NI$ be given with $|\operatorname{Supp} \alpha| = m > 0$ and suppose the result is known for all elements $\gamma \in NI$ of smaller support size. Choose $T \subset \operatorname{Supp} \alpha$ minimal with respect to the property that $NI \cap R * T \neq 0$. If $y \in T$, then $\operatorname{Supp} \alpha\bar{y}^{-1} = (\operatorname{Supp} \alpha)y^{-1} \supset Ty^{-1}$, Ty^{-1} also has this minimal property since NI is a right ideal, and $1 \in Ty^{-1}$. Since it clearly suffices to show that $D^{m+1}\alpha\bar{y}^{-1} \subset \bar{G}NI^\delta\bar{G}$, we can replace α by $\alpha\bar{y}^{-1}$ and T by Ty^{-1} and hence assume that $1 \in T$. Thus $1 \in \operatorname{Supp} \alpha$ and $T \in \mathcal{T}$.

Let $c = \operatorname{tr} \alpha$ be the identity coefficient of α and let $d \in D \subset A_T$. Then by definition of A_T there exists an element $\beta \in NI \cap R * T$ with $\operatorname{tr} \beta = d$. Thus $\gamma = d\alpha - \beta c \in NI$ and since $\operatorname{Supp} \beta \subset \operatorname{Supp} \alpha$ and $\operatorname{tr} \gamma = 0$, we have $|\operatorname{Supp} \gamma| < m$. By induction we deduce that $D^m\gamma \subset \bar{G}NI^\delta\bar{G}$ and hence $D^m d\alpha \subset \bar{G}NI^\delta\bar{G} + D^m\beta c$. Now we have shown above that $T \subset H$ and hence $\beta c \in I \cap R * H \subset I^\delta$. Thus, since $m \geqq 1$ and $D \subset N$ we have $D^m\beta c \subset NI^\delta \subset \bar{G}NI^\delta\bar{G}$. We conclude therefore that $D^m d\alpha \subset \bar{G}NI^\delta\bar{G}$ and since this holds for all $d \in D$ we have $D^{m+1}\alpha \subset \bar{G}NI^\delta\bar{G}$. The induction step is proved.

In particular, if $k = |G|$, we deduce from the above that $D^{k+1}NI \subset \bar{G}NI^\delta\bar{G}$. But observe that $D^{k+1}N \neq 0$, by Lemma 3.2 (i), since $D \subset N$ and $D \neq 0$. Thus if E is defined by

$$E = \{r \in R \mid rI \subset \bar{G}NI^\delta\bar{G}\}$$

then E is not zero because $E \supset D^{k+1}N \neq 0$. On the other hand, E is certainly a G-invariant ideal of R so we have an appropriate $E \neq 0$ with

$$EI \subset \bar{G}NI^\delta\bar{G} \subset I^{\delta\nu}$$

where the latter inclusion is of course trivial. This completes the proof.

Now $Q*H$ is an ideal of $R*H$ and we let

$$\kappa : R*H \to (R*H)/(Q*H) = (R/Q)*H = \tilde{R}*H$$

denote the natural homomorphism. Here $\tilde{R} = R/Q$ is of course a prime ring. If L is an ideal of $R*H$ containing $Q*H$ we let L^{κ} denote its image in $R*H$. If $\tilde{L}$ is an ideal of $\tilde{R}*H$ we let $\tilde{L}^{\kappa^{-1}}$ denote its complete inverse image in $R*H$. It is clear that the maps $^{\kappa}$ and $^{\kappa^{-1}}$ yield a one-to-one correspondence between these two sets of ideals. Observe that if I is an ideal of $R*G$, then $R*H \supset I^{\delta} \supset Q*H$ and hence $I^{\delta\kappa}$ is an ideal of $\tilde{R}*H$. On the other hand, if $\tilde{L}$ is an ideal of $\tilde{R}*H$, then $\tilde{L}^{\kappa^{-1}}$ is an ideal of $R*H$ and thus $\tilde{L}^{\kappa^{-1}\nu}$ is an ideal of $R*G$.

Recall from Section 2 that an ideal $\tilde{L}$ of $\tilde{R}*H$ is called $\tilde{R}$-cancelable if and only if $\tilde{A}\tilde{\alpha} \subset \tilde{L}$ for $\tilde{\alpha} \in \tilde{R}*H$ and $\tilde{A}$ a nonzero H-invariant ideal of $\tilde{R}$ implies that $\tilde{\alpha} \in \tilde{L}$. In the same sense we can speak of R-cancelable ideals in $R*G$. Thus the ideal I of $R*G$ is said to be R-cancelable if and only if for any $\alpha \in R*G$ and any nonzero G-invariant ideal A of R, $A\alpha \subset I$ implies that $\alpha \in I$.

LEMMA 3.5. *With the above notation we have*

(i) *If I is an ideal of $R*G$ with $I \cap R = 0$, then $I^{\delta\kappa} \cap \tilde{R} = 0$.*

(ii) *If $\tilde{L}$ is an ideal of $\tilde{R}*H$ with $\tilde{L} \cap \tilde{R} = 0$, then $\tilde{L}^{\kappa^{-1}\nu} \cap R = 0$.*

(iii) *If the ideal $\tilde{L}$ of $\tilde{R}*H$ is $\tilde{R}$-cancelable, then $\tilde{L}^{\kappa^{-1}\nu}$ is an R-cancelable ideal of $R*G$.*

(iv) *If $\tilde{L}_1, \tilde{L}_2, \cdots, \tilde{L}_m$ are $\tilde{R}$-cancelable ideals of $\tilde{R}*H$, then*

$$\bigcap_j \tilde{L}_j^{\kappa^{-1}\nu} = \Big(\bigcap_j \tilde{L}_j\Big)^{\kappa^{-1}\nu}.$$

PROOF. Let L be an ideal of $R*H$ containing $Q*H$. We show that $L^{\kappa} \cap \tilde{R} = 0$ if and only if $NL \cap R = 0$. Suppose first that $L^{\kappa} \cap \tilde{R} = 0$. If $\alpha \in NL \cap R$, then $\kappa(\alpha) \in L^{\kappa} \cap \tilde{R} = 0$ so $\alpha \in R \cap (Q*H) = Q$. But certainly $NL \cap R \subset (N*H) \cap R = N$ so $\alpha \in N \cap Q = 0$. Conversely let $NL \cap R = 0$ and let $\alpha \in L$ with $\kappa(\alpha) \in L^{\kappa} \cap \tilde{R}$. The latter implies that there exists $r \in R$ with $\kappa(\alpha) = \alpha + Q*H = r + Q*H$. Thus $\alpha - r \in Q*H$ and $N(\alpha - r) \subset N(Q*H) = 0$ so $N\alpha = Nr \in NL \cap R = 0$. But $Nr = 0$ implies that $r \in Q$ so $\kappa(\alpha) = \kappa(r) = 0$ and $L^{\kappa} \cap \tilde{R} = 0$.

Note also that by Lemma 3.1 (iii) we have $N\bar{G}N = N\bar{H}N$ and hence

$$N(\bar{G}NL\bar{G}) = (N\bar{H}N)L\bar{G} \subset NL\bar{G}.$$

Now let $\alpha \in R*G$ and write $\alpha = \Sigma_1^n \alpha_i \bar{x}_i$ where $\{1 = x_1, x_2, \cdots, x_n\}$ is a right transversal for H in G and where each $\alpha_i \in R * H$. If E is an ideal of R with $E\alpha \subset L^\nu$, then $ME\alpha \subset ML^\nu \subset \bar{G}NL\bar{G}$, by definition of L^ν, and hence

$$NME\alpha \subset N(\bar{G}NL\bar{G}) \subset NL\bar{G} = \sum_1^n NL\bar{x}_i.$$

Thus for each i we conclude that

$$NME\alpha_i \subset NL.$$

(i) Suppose $I \cap R = 0$. Since $NI^\delta \subset I$ we have $NI^\delta \cap R = 0$ and hence by the above, with $L = I^\delta$, we have $I^{\delta\kappa} \cap \tilde{R} = 0$.

(ii) Suppose $\tilde{L} \cap \tilde{R} = 0$ and set $L = \tilde{L}^{\kappa^{-1}}$. Then we know that $NL \cap R = 0$. Let $E = \tilde{L}^{\kappa^{-1}\nu} \cap R = L^\nu \cap R$ and let $\alpha = 1$ in the above so that surely $E\alpha \subset L^\nu$. Since $\alpha_1 = 1$ we therefore deduce that

$$NME = NME\alpha_1 \subset NL \cap R = 0.$$

But M and E are G-invariant ideals of R and $N, M \neq 0$. It therefore follows from the G-primeness of R that $E = 0$.

(iii) Now let $\tilde{L}$ be an $\tilde{R}$-cancelable ideal of $\tilde{R} * H$ and suppose that $E\alpha \subset \tilde{L}^{\kappa^{-1}\nu}$ where E is a nonzero G-invariant ideal of R and $\alpha \in R * G$. Write $\alpha = \Sigma_1^n \alpha_i \bar{x}_i$ as above. If $L = \tilde{L}^{\kappa^{-1}}$, then $E\alpha \subset L^\nu$ implies that

$$NME\alpha_i \subset NL \subset L$$

and hence $\tilde{A} \cdot \kappa(\alpha_i) \subset \tilde{L}$, where $\tilde{A}$ is the image of NME in $\tilde{R}$. Since Q does not contain N, M or E, $\tilde{A}$ is a nonzero H-invariant ideal of $\tilde{R}$ and thus our assumption on $\tilde{L}$ implies that $\kappa(\alpha_i) \in \tilde{L}$ for all i. In other words, $\alpha_i \in L$ for all i and we have

$$N\alpha \subset \sum_i N\alpha_i \bar{x}_i \subset NL\bar{G} \subset \bar{G}NL\bar{G}.$$

Moreover, since E is G-invariant, we see that for any $g \in G$

$$E\bar{g}\alpha = \bar{g}E\alpha \subset \tilde{L}^{\kappa^{-1}\nu}.$$

Thus the work of the preceding paragraph implies that $N\bar{g}\alpha \subset \bar{G}NL\bar{G}$ for all such $g \in G$ and it follows that

$$M\alpha = \sum \bar{g}^{-1}N\bar{g}\alpha \subset \bar{G}NL\bar{G}.$$

By definition, $\alpha \in L^\nu = \tilde{L}^{\kappa^{-1}\nu}$.

(iv) Suppose $\{\tilde{L}_j\}$ is a family of $\tilde{R}$-cancelable ideals of $\tilde{R} * H$ and let $L_j = \tilde{L}_j^{\kappa^{-1}}$. Then obviously, since $^\nu$ is order preserving, we have $(\bigcap_j L_j)^\nu \subset \bigcap_j L_j^\nu$. For the reverse inclusion take $\alpha = \Sigma_1^n \alpha_i \bar{x}_i \in \bigcap_j L_j^\nu$. Then for each j, $R\alpha \subset L_j^\nu$ and our above discussion, with $E = R$, shows that

$$NM\alpha_i \subset NL_j \subset L_j$$

for all i and j. Since the image of NM in $\tilde{R}$ is a nonzero H-invariant ideal of $\tilde{R}$, we can use the assumption on the ideals $\tilde{L}_j$ to deduce that $\alpha_i \in L_j$. Hence for all i we have

$$\alpha_i \in \bigcap_j L_j = L$$

and so $N\alpha \subset NL\bar{G} \subset \bar{G}NL\bar{G}$. Since α was arbitrary, we have shown that $N(\bigcap_j L_j^\nu) \subset \bar{G}NL\bar{G}$. But $F = \{r \in R \mid r(\bigcap_j L_j^\nu) \subset \bar{G}NL\bar{G}\}$ is surely a G-invariant ideal of R and since $F \supset N$ we conclude that $F \supset M$. In other words, we have shown that $M(\bigcap_j L_j^\nu) \subset \bar{G}NL\bar{G}$ and hence, by definition, we have

$$\bigcap_j L_j^\nu \subset L^\nu = \Big(\bigcap_j L_j\Big)^\nu.$$

Thus

$$\bigcap_j \tilde{L}_j^{\kappa^{-1}\nu} = \bigcap_j L_j^\nu = \tilde{L}^{\kappa^{-1}\nu}$$

where $\tilde{L} = (\bigcap_j L_j)^\kappa = \bigcap_j \tilde{L}_j$.

Note that $\tilde{R} = R/Q$ is a prime ring, so we know a good deal about the prime ideals of $\tilde{R} * H$ from the work of Section 2. The following is the main result of this section. We prove it simultaneously with Lemma 3.7.

THEOREM 3.6. *Let $R * G$ be a crossed product of the finite group G over the ring R. Assume R is G-prime and let Q be a minimal prime of R with H the stabilizer of Q in G. Then the maps $^{\delta\kappa}$ and $^{\kappa^{-1}\nu}$ yield a one-to-one correspondence between the prime ideals P of $R * G$ with $P \cap R = 0$ and the prime ideals $\tilde{L}$ of $\tilde{R} * H = (R * H)/(Q * H)$ with $\tilde{L} \cap \tilde{R} = 0$. More precisely:*

(i) *If P is a prime ideal of $R * G$ with $P \cap R = 0$, then $P^{\delta\kappa}$ is a prime ideal of $\tilde{R} * H$ with $P^{\delta\kappa} \cap \tilde{R} = 0$. Furthermore*

$$P = P^{\delta\kappa \cdot \kappa^{-1}\nu} = P^{\delta\nu}.$$

(ii) *Let $\tilde{L}$ be a prime ideal of $\tilde{R} * H$ with $\tilde{L} \cap \tilde{R} = 0$. Then $\tilde{L}^{\kappa^{-1}\nu}$ is a prime ideal of $R * G$ with $\tilde{L}^{\kappa^{-1}\nu} \cap R = 0$ and*

$$\tilde{L} = \tilde{L}^{\kappa^{-1}\nu \cdot \delta\kappa}.$$

Lemma 3.7. *Let $R * G$ be given with R a G-prime ring. If P is a prime ideal of $R * G$ with $P \cap R = 0$ and if I is an ideal of $R * G$ properly containing P, then $I \cap R \neq 0$.*

Proof. (i) Let P be a prime ideal of $R * G$ with $P \cap R = 0$. By Lemma 3.5 (i) we know at least that $P^{\delta\kappa} \cap \tilde{R} = 0$. Set $L = P^{\delta}$ and suppose that L_1 and L_2 are ideals of $R * H$ with $L \supset L_1 L_2$. Then Lemmas 3.4 (ii) and 3.3 (ii) yield

$$P \supset MP^{\delta\nu} \supset M(L_1L_2)^{\nu} \supset ML_1^{\nu}ML_2^{\nu} = (M * G)L_1^{\nu}(M * G)L_2^{\nu}.$$

Hence since P is prime and $P \not\supset M * G$, we conclude that $P \supset L_i^{\nu}$ for some i. Thus, by Lemma 3.4 (i),

$$L = P^{\delta} \supset L_i^{\nu\delta} \supset L_i$$

and we have shown that L is a prime ideal of $R * H$. Therefore since $L \supset Q * H$ we conclude that $\tilde{L} = L^{\kappa} = P^{\delta\kappa}$ is a prime ideal of $\tilde{R} * H$.

We now compare P with $P^{\delta\nu} = P^{\delta\kappa \cdot \kappa^{-1}\nu}$. First, Lemma 3.4 (ii) yields $P \supset MP^{\delta\nu} = (M * G)P^{\delta\nu}$. Thus since P is prime and $P \not\supset M * G$ we have $P \supset P^{\delta\nu}$. In the other direction, Lemma 3.4 (ii) also implies that there exists a nonzero G-invariant ideal E of R with $EP \subset P^{\delta\nu} = \tilde{L}^{\kappa^{-1}\nu}$. But $\tilde{L}$ is prime and $\tilde{L} \cap \tilde{R} = 0$ so $\tilde{L}$ is $\tilde{R}$-cancelable and we conclude from Lemma 3.5 (iii) that $P^{\delta\nu}$ is R-cancelable. Thus $EP \subset P^{\delta\nu}$ implies that we have the reverse inclusion $P \subset P^{\delta\nu}$ so $P = P^{\delta\nu}$ and (i) is proved.

(ii) Now let $\tilde{L}$ be a prime ideal of $\tilde{R} * H$ with $\tilde{L} \cap \tilde{R} = 0$ and set $L = \tilde{L}^{\kappa^{-1}}$. Then by Lemma 3.5 (ii) we have $L^{\nu} \cap R = \tilde{L}^{\kappa^{-1}\nu} \cap R = 0$. Now suppose I is any ideal of $R * G$ with $I \supset L^{\nu}$ and $I \cap R = 0$. Then Lemmas 3.4 (i) and 3.5 (i) yield

$$I^{\delta\kappa} \supset L^{\nu\delta\kappa} \supset L^{\kappa} = \tilde{L}$$

and $I^{\delta\kappa} \cap \tilde{R} = 0$. Since $\tilde{R} = R/Q$ is a prime ring, we can now apply Lemma 2.6 to the crossed product $\tilde{R} * H$ to conclude that $I^{\delta\kappa} = \tilde{L}$ or equivalently that $I^{\delta} = L$. Furthermore, by Lemma 3.4 (ii), there exists a nonzero G-invariant ideal E of R with

$$EI \subset I^{\delta\nu} = L^{\nu} = \tilde{L}^{\kappa^{-1}\nu}.$$

Hence since $\tilde{L}$ is $\tilde{R}$-cancelable, Lemma 3.5 (iii) implies that L^{ν} is R-cancelable and $EI \subset L^{\nu}$ yields $I \subset L^{\nu}$. Thus $I = L^{\nu}$.

In particular since $L^{\nu} \cap R = 0$, we can take $I = L^{\nu}$ in the above and deduce that

$$\tilde{L}^{\kappa^{-1}\nu\cdot\delta\kappa} = L^{\nu\delta\kappa} = I^{\delta\kappa} = \tilde{L}.$$

Next suppose that I_1 and I_2 are ideals of $R * G$ properly containing L^{ν}. Then by the above we have $I_1 \cap R \neq 0$, $I_2 \cap R \neq 0$ and the G-primeness of R yields

$$I_1I_2 \cap R \supset (I_1 \cap R)(I_2 \cap R) \neq 0.$$

Thus using $L^{\nu} \cap R = 0$, we conclude that $I_1I_2 \not\subseteq L^{\nu}$ and therefore L^{ν} is a prime ideal of $R * G$. This completes the proof of Theorem 3.6.

Finally for Lemma 3.7, let P be a prime ideal of $R * G$ with $P \cap R = 0$ and let $I \supsetneqq P$. Then by (i) above $P = L^{\nu} = \tilde{L}^{\kappa^{-1}\nu}$ where $L = P^{\delta}$ and $\tilde{L}$ is a prime ideal of $\tilde{R} * H$ with $\tilde{L} \cap \tilde{R} = 0$. But we have just shown, for such ideals L, that if $I \supsetneqq L^{\nu}$ then $I \cap R \neq 0$. Thus Lemma 3.7 is also proved.

It is clear that we can now combine the maps ${}^{\delta\kappa}$ and ${}^{\kappa^{-1}\nu}$ with the maps ^{d} and ^{u} of Theorem 2.5, applied to $\tilde{R} * H$, to obtain a one-to-one correspondence between the prime ideals P of $R * G$ with $P \cap R = 0$ and the H-prime ideals of a certain twisted group algebra $C^t[H_{\mathrm{inn}}]$ where C is the extended centroid of $\tilde{R}$. However, instead of formalizing this further, we will content ourselves with proving the main results of this paper, namely Theorems 1.2 and 1.3.

Proof of Theorem 1.2. In view of the comments of Section 1, this is an immediate consequence of Lemma 3.7.

For Theorem 1.3 we require just one more lemma, given below, which proves that a certain ideal of $R * G$ is nilpotent. Unfortunately the multiplication formula of Lemma 3.3 (ii) is not sufficient to do this because of the presence of the factor M which occurs within the product. Thus the argument given here is a good deal less straightforward than the analogous proof in Section 2.

Lemma 3.8. *If $\tilde{J}$ denotes the unique maximal nilpotent ideal of $\tilde{R} * H$, as given by Lemma 2.7, then $\tilde{J}^{\kappa^{-1}\nu}$ is nilpotent of degree $\leqq |H|$.*

Proof. Set $J = \tilde{J}^{\kappa^{-1}}$. We first show that for any $\alpha \in \bar{G}NJ\bar{G}$ there exists an H-invariant ideal I of R properly containing Q such that

$$NIN\bar{G}\alpha \subset NJN \cdot M\bar{G}.$$

Since $\alpha \in \bar{G}NJ\bar{G}$ we have $\alpha \in \Sigma_i \bar{G}N\alpha_i\bar{G}$, a finite sum, with $\alpha_i \in J$. By Lemma

2.8 (ii), applied to the nonzero ideal $\tilde{N}\tilde{M}$ of $\tilde{R}$, there exists a nonzero H-invariant ideal $\tilde{I}$ of $\tilde{R}$ such that $\tilde{I}\cdot\kappa(\alpha_i)\subset\tilde{J}\tilde{N}\tilde{M}$. Thus, if I denotes the inverse image of $\tilde{I}$ in R, then I is also H-invariant and we have

$$I\alpha_i \subset JNM + Q * H.$$

Let $K_i = \bar{G}N\alpha_i\bar{G}$ and consider $(NIN\bar{G})K_i$. Since $N\bar{G}N = N\bar{H}N$, by Lemma 3.1 (iii), and since NIN is H-invariant, we have

$$\begin{aligned}(NIN\bar{G})K_i &= NI(N\bar{G}N)\alpha_i\bar{G}\\ &= NI(N\bar{H}N)\alpha_i\bar{G}\\ &= \bar{H}NIN^2\alpha_i\bar{G}\\ &\subset \bar{H}NI\alpha_i\bar{G}.\end{aligned}$$

But $I\alpha_i \subset JNM + Q * H$ so $NI\alpha_i \subset NJNM$ and therefore

$$\begin{aligned}(NIN\bar{G})K_i &\subset \bar{H}(NI\alpha_i)\bar{G}\\ &\subset \bar{H}(NJNM)\bar{G}\\ &= NJNM\bar{G}\end{aligned}$$

where the latter occurs since the factor $\bar{H}$ can be absorbed into the ideal NJ of $R * H$. Hence, since $\alpha \in \Sigma_i K_i$, we conclude finally that

$$NIN\bar{G}\alpha \subset \sum_i (NIN\bar{G})K_i \subset NJNM\bar{G}.$$

By Lemma 2.8 (i), there exists a sequence $\tilde{J} = \tilde{J}_1, \tilde{J}_2, \cdots$ of $\tilde{R}$-cancelable ideals in $\tilde{R} * H$ such that $\tilde{J}\tilde{J}_i \subset \tilde{J}_{i+1}$ and $\tilde{J}_k = 0$ for some $k \leqq |H|$. Set $J_i = \tilde{J}_i^{\kappa^{-1}}$ so that $J_1 = J$, $JJ_i \subset J_{i+1}$ and $J_k = Q * H$. We show now that $J^\nu J_i^\nu \subset J_{i+1}^\nu$. First let $\alpha \in \bar{G}NJ\bar{G}$ and let $\beta \in J_i^\nu$. Then $M\bar{G}\beta = \bar{G}M\beta \subset \bar{G}NJ_i\bar{G}$. Thus if I denotes the ideal constructed for α in the first paragraph of the proof, then we have from the above and Lemma 3.3 (i).

$$\begin{aligned}NIN\bar{G}\alpha\beta &\subset NJNM\bar{G}\beta\\ &\subset NJN\bar{G}NJ_i\bar{G}\\ &\subset NJJ_i\bar{G}\\ &\subset (JJ_i)^\nu.\end{aligned}$$

Now set $E = \Sigma_{x \in G} (NIN)^{\bar{x}}$. Then E is a G-invariant ideal of R which is certainly nonzero, since $NIN \not\subset Q$, and the above shows that we have

$$E\alpha\beta \subset (JJ_i)^{\nu} \subset J_{i+1}^{\nu}.$$

However, $\tilde{J}_{i+1}$ is $\tilde{R}$-cancelable so $J_{i+1}^{\nu} = \tilde{J}_{i+1}^{\kappa^{-1}\nu}$ is R-cancelable, by Lemma 3.5 (iii), and hence we deduce from $E\alpha\beta \subset J_{i+1}^{\nu}$ that

$$\alpha\beta \in J_{i+1}^{\nu}.$$

We have therefore shown that

$$(\bar{G}NJ\bar{G})J_i^{\nu} \subset J_{i+1}^{\nu}.$$

But observe that $MJ^{\nu} \subset \bar{G}NJ\bar{G}$. Thus we conclude from this that

$$MJ^{\nu}J_i^{\nu} \subset (\bar{G}NJ\bar{G})J_i^{\nu} \subset J_{i+1}^{\nu}$$

and again using the fact that J_{i+1}^{ν} is R-cancelable we deduce that $J^{\nu}J_i^{\nu} \subset J_{i+1}^{\nu}$.

Finally suppose that $(J^{\nu})^i \subset J_i^{\nu}$ for some i. Then, by what we have just shown, we have

$$(J^{\nu})^{i+1} = J^{\nu}(J^{\nu})^i \subset J^{\nu}J_i^{\nu} \subset J_{i+1}^{\nu}.$$

Thus since $J = J_1$, it now follows by induction that $(J^{\nu})^i \subset J_i^{\nu}$ for all i. But $J_k = Q * H$ so $NJ_k = 0$ and $J_k^{\nu} = 0$. Hence $(J^{\nu})^k = 0$ and the lemma is proved since $k \leqq |H|$.

We now offer the promised

Proof of Theorem 1.3. We are given a crossed product $R * G$ with G finite and with R a G-prime ring. As usual let Q be a minimal prime of R, given by Lemma 3.1 (i), and let H be the stabilizer of Q in G. By Lemma 2.7, applied to the crossed product $\tilde{R} * H$, we see that there are finitely many primes $\tilde{L}$ of $\tilde{R} * H$ with $\tilde{L} \cap \tilde{R} = 0$. Indeed if these are $\tilde{L}_1, \tilde{L}_2, \cdots, \tilde{L}_n$, then $n \leqq |H| \leqq |G|$ and $\tilde{J} = \tilde{L}_1 \cap \tilde{L}_2 \cap \cdots \cap \tilde{L}_n$ is the unique largest nilpotent ideal of $\tilde{R} * H$. For each i, set $P_i = \tilde{L}_i^{\kappa^{-1}\nu}$. Then we conclude from Theorem 3.6 that $P_1, P_2, \cdots, P_n$ are the unique prime ideals of $R * G$ having trivial intersection with R.

Since each $\tilde{L}_j$ is prime and satisfies $\tilde{L}_j \cap \tilde{R} = 0$, we see that each $\tilde{L}_j$ is $\tilde{R}$-cancelable. Thus Lemma 3.5 (iv) yields

$$P_1 \cap P_2 \cap \cdots \cap P_n = \bigcap_j \tilde{L}_j^{\kappa^{-1}\nu} = \Big(\bigcap_j \tilde{L}_j \Big)^{\kappa^{-1}\nu} = \tilde{J}^{\kappa^{-1}\nu}$$

and we deduce from Lemma 3.8 that $P_1 \cap P_2 \cap \cdots \cap P_n$ is nilpotent of degree $\leqq |H|$ and hence of degree $\leqq |G|$. This of course implies that $P_1 \cap P_2 \cap \cdots \cap P_n$ is the unique largest nilpotent ideal of $R * G$.

Finally let P be any prime ideal of $R * G$. Then P contains the nilpotent ideal $P_1 \cap P_2 \cap \cdots \cap P_n$ and hence $P \supseteq P_i$ for some i. This shows that the minimal primes of $R * G$ are the minimal members of the set $\{P_1, P_2, \cdots, P_n\}$. But observe that $P_i \supseteq P_j$ implies by Theorem 3.6 that $\tilde{L}_i = P_i^{\delta\kappa} \supseteq P_j^{\delta\kappa} = \tilde{L}_j$ and hence since $\tilde{L}_i$ is minimal, we must have $i = j$. This shows that $P_1, P_2, \cdots, P_n$ are precisely the minimal primes of $R * G$ and the theorem is proved.

We close this section with two corollaries. The first explains why the condition of G-semiprimeness of $R * H$ can be replaced by the simpler condition of ordinary semiprimeness in [8, theorem 2.7]. The second shows by example that the one-to-one correspondences given here actually offer much more information than we state in the main theorems.

Corollary 3.9. *Let $R * G$ be a crossed product with G finite and with R a semiprime ring. Then $R * G$ has a unique largest nilpotent ideal J and $J^{|G|} = 0$.*

Proof. Let J be the sum of all nilpotent ideals of $R * G$. We show below that $J^{|G|} = 0$ and from this it will follow immediately that J is the unique largest nilpotent ideal of $R * G$.

Observe that if P is a prime ideal of R, then $\bigcap_{x \in G} P^x$ is clearly a G-prime ideal. Hence since R is semiprime, it follows that the intersection of all G-prime ideals of R is zero. Now let Q be a G-prime ideal of R and let

$$\bar{\ }: R * G \to R * G / Q * G \simeq (R/Q) * G$$

denote the natural homomorphism. Then $\bar{J}$ is a sum of nilpotent ideals of $(R/Q) * G$ and R/Q is a G-prime ring. Thus we deduce from Theorem 1.3 (iii) that $\bar{J}$ is nilpotent and in fact that $\bar{J}^{|G|} = 0$. In other words, $J^{|G|} \subseteq Q * G$ and, since this holds for all such Q, we have

$$J^{|G|} \subseteq \bigcap_Q Q * G = \Big(\bigcap_Q Q\Big) * G = 0.$$

The result follows.

Corollary 3.10. *Let $p > 0$ be a prime and let $R * G$ be given with G a finite p-group and with R a G-prime ring of characteristic p. Then $R * G$ has a unique minimal prime ideal.*

PROOF. In view of Theorem 1.3, P is a minimal prime ideal of $R * G$ if and only if $P \cap R = 0$. Suppose first that R is a prime ring. Then by Theorem 2.5, the number of minimal primes of $R * G$ is the same as the number of G-prime ideals of $E = C^t[G_{inn}]$. But observe that C is a field of characteristic p and G_{inn} is a finite p-group, so E has a unique prime ideal by [8, lemma 3.3 (i)]. Hence E has a unique G-prime ideal and $R * G$ has a unique minimal prime in this case.

Finally, if R is G-prime we apply Theorem 3.6 and its notation. Thus the number of minimal primes of $R * G$ is the same as the number of minimal primes of $(R/Q) * H$. Since R/Q is a prime ring of characteristic p and H is a finite p-group, the result follows from the prime case considered above.

§4. Chains of prime and primitive ideals

In addition to the ring extension $R \subset R * G$ studied in the preceding sections, we shall now consider another type of ring extension. Namely, suppose G is a finite group acting as automorphisms on a ring R. Then $R^G = \{r \in R \mid r^g = r$ for all $g \in G\}$ is a subring of R, the so-called fixed subring of R. We will be concerned here with some aspects of the relationship between R and R^G. More precisely, for both types of extensions, $R \subset R * G$ and $R^G \subset R$, we shall show how to pass from a chain of prime ideals in one of the rings to a chain of prime ideals in the other ring having the same length. Here of course the length of the chain

$$P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n$$

of distinct prime ideals P_i of a ring is defined to be the number n. Special attention will be given to the primitive ideals, that is to those ideals that are annihilators of simple right modules.

We start with the extension $R \subset R * G$ where the results we have obtained so far can be easily applied. As we will see, little additional work is required to distinguish the primitive ideals among the primes. Thus in the first part of this section, R will be a ring (with 1), G will be a finite group and $R * G$ will denote a crossed product of G over R.

If A is an ideal of any ring S, then a minimal covering prime of A is, by definition, a prime ideal P of S containing A such that P/A is a minimal prime of the ring S/A. The following two lemmas describing the minimal covering primes of certain ideals in $R * G$ and in R, contain somewhat more detail than is actually needed here.

LEMMA 4.1. *Let A be an ideal of R and set*

$$A' = \Big(\bigcap_{x\in G} A^{\bar{x}}\Big) R * G = \Big(\bigcap_{x\in G} A^{\bar{x}}\Big) * G.$$

Then A' is an ideal of $R * G$ with $A' \cap R = \bigcap_{x\in G} A^{\bar{x}}$. Moreover

(i) *If A is prime, then A' has finitely many minimal covering primes. If these are $P_1, P_2, \cdots, P_n$, then $P_i \cap R = \bigcap_{x\in G} A^{\bar{x}}$ for all i and $\bigcap_1^n P_i$ is nilpotent modulo A'.*

(ii) *If A is primitive, then the minimal covering primes of A' are also primitive.*

(iii) *If A is maximal, then so are the minimal covering primes of A'.*

PROOF. Since the first assertions are obvious we proceed to verify (i), (ii) and (iii).

(i) If A is prime, then, by the introductory remarks of Section 3, $\tilde{R} = R/\bigcap_{x\in G} A^{\bar{x}}$ is a G-prime ring. Furthermore the minimal covering primes of A' correspond to the minimal primes of the ring $(R * G)/A' \simeq \tilde{R} * G$ which are described in Theorem 1.3. In particular, they are finite in number, all of them satisfy $\tilde{P} \cap \tilde{R} = 0$, and their intersection is the unique largest nilpotent ideal of $\tilde{R} * G$. Assertion (i) follows immediately from this.

(ii) Now suppose A is primitive and let V be an irreducible right R-module with annihilator A. Consider the induced $R * G$-module $W = V \otimes_R R * G$. As an R-module, W can be written as a finite direct sum

$$W_R = \bigoplus \sum_{x\in G} V \otimes \bar{x}$$

of the R-submodules $V \otimes \bar{x}$. The latter are of course conjugate to V and hence they are also irreducible. In particular W, as an R-module, has a composition series of finite length and *a fortiori* W has a finite composition series, say

$$W = W_0 \supsetneqq W_1 \supsetneqq \cdots \supsetneqq W_t = 0$$

as an $R * G$-module. It is easy to see that the annihilator of W is precisely equal to $(\bigcap_{x\in G} (\mathrm{ann}_R V)^{\bar{x}}) * G = A'$.

Now let $P_i = \mathrm{ann}_{R*G}(W_{i-1}/W_i)$ be the annihilator of the irreducible $R * G$-module W_{i-1}/W_i. Then the ideals P_i are primitive and certainly $P_i \supset A' = \mathrm{ann}_{R*G} W$. Moreover, the product $P_1 \cdot P_2 \cdot \cdots \cdot P_t$ clearly annihilates the module W so we have $P_1 \cdot P_2 \cdot \cdots \cdot P_t \subset A'$. It follows from this that any prime containing A' contains one of the ideals P_i. In particular the minimal covering primes of A' form a subset of $\{P_1, P_2, \cdots, P_t\}$ and hence are primitive.

(iii) Finally let A be maximal and let P be a minimal covering prime of A'. Suppose by way of contradiction that P is properly contained in an ideal I of $R*G$. Then, by Theorem 1.2, it follows that $I \cap R \supsetneqq P \cap R = \bigcap_{x\in G} A^{\bar{x}}$, where the latter equality holds by part (i). Choose a maximal ideal B of R containing $I \cap R$. Then $\bigcap_{x\in G} A^{\bar{x}} \subset B$ and hence $A^{\bar{y}} \subset B$ for some $y \in G$. Thus the maximality of A implies that equality must hold here so $B = A^{\bar{y}}$. Since $I \cap R$ is a G-invariant ideal of R and $I \cap R \subset B$, we conclude that $I \cap R \subset \bigcap_{x\in G} B^{\bar{x}} = \bigcap_{x\in G} A^{\bar{x}} = P \cap R$. This contradiction shows that P is in fact maximal.

The next lemma deals with the reverse process of passing from an ideal of $R*G$ to an ideal of R.

LEMMA 4.2. *Let I be an ideal of $R*G$. Then $I \cap R$ is a G-invariant ideal of R. Moreover*

(i) *If I is semiprime, then $I \cap R$ is also semiprime.*

(ii) *If I is prime, then $I \cap R = \bigcap_{x\in G} P^{\bar{x}}$ for some prime ideal P of R which is unique up to G-conjugacy.*

(iii) *If I is primitive, then the ideal P in* (ii) *is also primitive.*

PROOF. The first assertion is clear.

(i) Let I be semiprime and let A be an ideal of R which is nilpotent modulo $I \cap R$. Then the G-invariant ideal $B = \Sigma_{x\in G} A^{\bar{x}}$ is also nilpotent modulo $I \cap R$, since G is finite. Observe that $B*G$ is an ideal of $R*G$ satisfying $(B*G)^i = B^i*G$ for all i. Thus some power of $B*G$ is contained in $(I\cap R)*G$ and hence in I. The assumption on I now implies that $B*G \subset I$ and hence $A \subset B \subset I \cap R$. This proves that $I \cap R$ is semiprime.

(ii) This follows by quoting earlier results. Indeed, Lemma 1.1 says that $I \cap R$ is a G-prime ideal of R and hence Lemma 3.1 (i) yields the existence of P. Finally, the uniqueness of P, up to G-conjugacy, is immediate from Lemma 3.1 (ii).

(iii) Now suppose I is primitive and let V be an irreducible right $R*G$-module with annihilator I. Then an appropriate version of Clifford's classical restriction theorem (see [6, lemma 1.3]) shows that the restricted module V_R contains an irreducible submodule W and in fact can be written as

$$V_R = \sum_{x\in G} W\bar{x}.$$

Let Q denote the annihilator of W so that Q is a primitive ideal in R. Then $Q^{\bar{x}}$ is the annihilator of the R-submodule $W\bar{x}$ of V_R and hence

$$I \cap R = \operatorname{ann}_R V = \bigcap_{x \in G} \operatorname{ann}_R W\bar{x} = \bigcap_{x \in G} Q^x.$$

The uniqueness of P now implies that $P = Q^y$ for some $y \in G$, so P is primitive.

We now combine these two lemmas to obtain Theorem 4.4, a result which extends [6, theorem 1.7]. Recall from the introduction that a ring S is said to have prime rank n if S has a chain of prime ideals of length n but no longer such chain. If there is no bound for the lengths of chains of prime ideals in S, then of course the prime rank of S is infinite. The primitive rank of S is defined analogously via primitive ideals. Thus the primitive rank of S is 0 if and only if all primitive ideals of S are maximal. For later work, it is convenient to first obtain the following.

LEMMA 4.3. *Let $e \in R * G$ be an idempotent with* tr e, *the identity coefficient of e, a unit in R. If R has prime (or primitive) rank $\geqq n$, then there exists a chain*

$$P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n$$

of prime (or primitive) ideals of $R * G$ such that $e \notin P_n$.

PROOF. By assumption there exists a chain

$$Q_0 \subsetneqq Q_1 \subsetneqq \cdots \subsetneqq Q_n$$

of prime (or primitive) ideals of R. Observe that, for each i, $\bigcap_{x \in G} Q_i^x \subset \bigcap_{x \in G} Q_{i+1}^x$ and in fact these two ideals are distinct. Indeed, if equality occurred, it would follow that $Q_i \supset \bigcap_{x \in G} Q_{i+1}^x$ and thus $Q_i \supset Q_{i+1}^y \supsetneqq Q_i^y$ for some $y \in G$. However, as we have already remarked in the proof of Lemma 3.1 (ii), a strict inclusion $Q_i \supsetneqq Q_i^y$ is impossible since G is finite.

Now observe that $e \notin Q_n' = (\bigcap_{x \in G} Q_n^x) * G$ since tr e, being a unit, is not contained in $\bigcap_{x \in G} Q_n^x$. Furthermore, since e is an idempotent, e cannot even be nilpotent modulo Q_n'. It therefore follows immediately from Lemma 4.1 (i) that there exists a minimal covering prime P_n of Q_n' such that $e \notin P_n$. We now successively apply the Going Down theorem (Theorem 1.3) and find prime ideals $P_{n-1}, P_{n-2}, \cdots, P_0$ of $R * G$ such that $P_{i+1} \supset P_i$, P_i is a minimal covering prime of Q_i' and $P_i \cap R = \bigcap_{x \in G} Q_i^x$. Then we have $e \notin P_n$ and

$$P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n$$

where the inequalities occur since

$$P_i \cap R = \bigcap_{x \in G} Q_i^x \neq \bigcap_{x \in G} Q_{i+1}^x = P_{i+1} \cap R.$$

Thus the result follows in the prime case. Finally, if all Q_i are assumed to be primitive, then each P_i is also primitive, by Lemma 4.1 (ii), and the lemma is proved.

THEOREM 4.4. *Let $R * G$ be a crossed product of the finite group G over the ring R. Then the prime rank of $R * G$ is equal to the prime rank of R and the primitive rank of $R * G$ is equal to the primitive rank of R.*

PROOF. By taking $e = 1$ in the preceding lemma, we deduce immediately that the prime rank and the primitive rank of R are at most equal to the corresponding ranks of $R * G$. For the reverse inequalities, suppose that

$$P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n$$

is a chain of prime (or primitive) ideals of $R * G$. Then by Theorem 1.2 the intersections $P_i \cap R$ are all distinct and by Lemma 4.2 (ii) we may write $P_i \cap R = \bigcap_{x \in G} Q_i^{\bar{x}}$ for a suitable prime ideal Q_i of R unique up to G-conjugation. Now fix Q_n. If Q_{i+1} is given, then since $Q_{i+1} \supset \bigcap_{x \in G} Q_i^{\bar{x}}$, we may certainly choose Q_i so that $Q_{i+1} \supset Q_i$. Thus by successively choosing $Q_{n-1}, Q_{n-2}, \cdots, Q_0$ we obtain the chain

$$Q_0 \subsetneqq Q_1 \subsetneqq \cdots \subsetneqq Q_n$$

where the inequalities occur since

$$\bigcap_{x \in G} Q_i^{\bar{x}} = P_i \cap R \subsetneqq P_{i+1} \cap R = \bigcap_{x \in G} Q_{i+1}^{\bar{x}}.$$

Thus we see that the prime rank of $R * G$ is at most equal to the prime rank of R and hence the two ranks must be equal. Finally, if each P_i is assumed to be primitive, then each Q_i is also primitive by Lemma 4.2 (iii).

We remark that the above argument actually shows that corresponding prime (or primitive) ideals of R and of $R * G$ have the same height.

We now turn to the pair $R^G \subset R$. Thus for the remainder of this section we assume that G is a finite group acting on the ring R. Using this action, we may then form the skew group ring RG. Here the elements of RG are of course formal sums $\Sigma_{x \in G} r_x \bar{x}$ with $r_x \in R$, addition in RG is defined componentwise and multiplication is defined distributively by means of the rule

$$(r_x \bar{x})(r_y \bar{y}) = r_x r_y^{x^{-1}} \overline{xy}.$$

In other words, RG is just the special case of a crossed product $R * G$ where the factor set is identically 1 and where the automorphisms $^{\bar{x}}$ are given by the action of G on R. Therefore our previous results apply to RG.

For the remainder of this section we also assume that $|G|^{-1} \in R$ so that RG contains the element $e = |G|^{-1}\Sigma_{x\in G}\bar{x}$. Observe that $\bar{y}e = e = e\bar{y}$ for all $y \in G$ and consequently e is an idempotent of RG. Moreover, for $r \in R$ we have

$$\begin{aligned} ere &= \left(|G|^{-1} \sum_{x\in G} \bar{x}\right) re \\ &= |G|^{-1}\left(\sum_{x\in G} r^{x^{-1}}\bar{x}e\right) \\ &= |G|^{-1}\left(\sum_{x\in G} r^{x^{-1}}e\right) \\ &= \left(|G|^{-1} \sum_{x\in G} r^{x}\right) e \\ &= t_G(r)e \end{aligned}$$

where $t_G(r) = |G|^{-1}\Sigma_{x\in G}r^x$ is obviously an element of R^G. Therefore, for any $r \in R$ and $x \in G$ we have $e(r\bar{x})e = er(\bar{x}e) = ere = t_G(r)e \in R^Ge$ and hence $eRGe \subset R^Ge$. On the other hand, since the elements of R^G obviously commute with e, we have $R^Ge = eR^G = eR^Ge \subset eRGe$ and thus

$$eRGe = eR^G = R^Ge \simeq R^G$$

where the latter isomorphism $R^G \simeq R^Ge$ is given by $r \to re$. In other words, R^G is related to RG in a natural way. We can then use our previous knowledge on the extension $R \subset RG$, together with well known and classical results on extensions of the type $eRGe \subset RG$, to derive information about the pair $R^G \subset R$ quite easily. Let us first summarize what is needed about the extension $eRGe \subset RG$ or, more generally, $fSf \subset S$ where f is any nonzero idempotent of the ring S. All this is of course well known and most of it can be found in Jacobson's book [4]. However, for the sake of completeness, we include the easy proof.

LEMMA 4.5. *Let f be a nonzero idempotent of the ring S. Then the map $\varphi : P \to fPf = P \cap fSf$ sets up a one-to-one correspondence between the set $\mathcal{S}_f$ of prime ideals of S not containing f and the set of all prime ideals of fSf. Moreover, if P, P_1 and P_2 are in $\mathcal{S}_f$, then $P_1^\varphi \subset P_2^\varphi$ if and only if $P_1 \subset P_2$ and P^φ is primitive if and only if P is primitive.*

PROOF. Observe that if A is an ideal of fSf, then SAS is an ideal of S with $SAS \cap fSf = f(SAS)f = (fSf)A(fSf) = A$.

Suppose that P is a prime ideal of S not containing f. We show that $P^\varphi = fPf$ is prime in fSf. First since $f \not\in P$ we have $f \not\in P^\varphi$ and hence P^φ is a proper ideal of fSf. Now suppose that A_1 and A_2 are ideals of fSf with $A_1A_2 \subset fPf$. Then

$$(SA_1S)(SA_2S) = SA_1(fSf)A_2S \subset SA_1A_2S \subset P$$

and we conclude from the primeness of P that $SA_iS \subset P$ for some i. Furthermore, for this i,

$$A_i = f(SA_iS)f \subset fPf$$

and we deduce that $fPf = P^\varphi$ is indeed a prime ideal of fSf.

Now let $P_1, P_2 \in \mathcal{S}_f$ and assume that $P_1^\varphi \subset P_2^\varphi$. Then $fP_1f \subset fP_2f$ and we have $(SfS)P_1(SfS) \subset P_2$. Hence since $f \not\in P_2$, we conclude from the primeness of P_2 that $P_1 \subset P_2$. Conversely, if $P_1 \subset P_2$ then obviously $P_1^\varphi = fP_1f \subset fP_2f = P_2^\varphi$. Thus we have shown that $P_1^\varphi \subset P_2^\varphi$ holds if and only if $P_1 \subset P_2$. In particular, we see immediately that φ is injective.

Next we show that φ is onto. Thus let Q be a prime ideal of fSf and observe, as above, that the ideal SQS of S satisfies $SQS \cap fSf = Q$. Hence, by Zorn's lemma, we may choose an ideal P of S which is maximal with respect to $P \cap fSf = Q$ and then it follows easily that P is prime. Furthermore, $f \not\in P$ so $P \in \mathcal{S}_f$. Since $P^\varphi = Q$ holds by the choice of P, we have therefore shown that φ is onto.

It remains to verify the assertions concerning primitive ideals. Thus let P be a primitive ideal of S with $f \not\in P$ and, say, P is the annihilator of the irreducible right S-module V. Then Vf is a right fSf-module which is nonzero, since $f \not\in P$. In fact Vf is irreducible. Indeed, if $U \subset Vf$ is a nonzero fSf-submodule of Vf, then it follows from the irreducibility of V that $US = V$ and hence $Vf = USf = U(fSf) = U$. Furthermore, the annihilator of Vf in fSf is easily seen to be $P^\varphi = fPf$. Thus, if P is primitive, then P^φ is primitive.

Conversely, let Q be a primitive ideal of fSf and let W be an irreducible right fSf-module with annihilator Q. Write $W \simeq fSf/X$ for some maximal right ideal X of fSf. Then $X_1 = XS \oplus (1-f)S$ is a right ideal of S with

$$X_1 \cap fSf = XS \cap fSf = XSf = X$$

and thus X_1 is contained in a maximal right ideal Y of S. Clearly, $Y \cap fSf = X$ since X is maximal and $f \not\in Y$ and hence, as fSf-modules, we have

$$V = S/Y \supset (fSf + Y)/Y \simeq fSf/X \simeq W.$$

Consequently, if P denotes the annihilator of the irreducible right S-module V, then P is a primitive ideal of S such that

$$P \cap fSf = fPf \subset \operatorname{ann}_{fSf} W = Q.$$

In particular, $f \notin P$. To prove the reverse inclusion, observe that $fSQ = (fSf)Q \subset X \subset Y$ and $(1-f)SQ \subset (1-f)S \subset Y$. Thus $SQ = fSQ + (1-f)SQ \subset Y$ and we see that SQS is a two sided ideal of S contained in Y. Thus $SQS \subset \operatorname{ann}_S S/Y = P$ and hence $Q \subset fPf = P^{\varphi}$. This proves the equality $Q = P^{\varphi}$ and we have shown that if Q is a primitive ideal of fSf, then $Q^{\varphi^{-1}}$ is also primitive.

Let us note the following simple consequence of Lemma 4.5. Namely, if f is a nonzero idempotent of S, then the prime rank and the primitive rank of fSf are bounded above by the corresponding ranks of S. Indeed, if $Q_1 \subsetneqq Q_2 \subsetneqq \cdots \subsetneqq Q_n$ is a chain of prime (or primitive) ideals of fSf, then by applying the map φ^{-1} as given above, we obtain a chain $P_1 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n$ of prime (or primitive) ideals of S. We now combine all these observations to obtain the

Proof of Theorem 1.4. For convenience let us use rank S to denote either the prime or primitive rank of S, whichever is relevant. Now we are given a finite group G acting on a ring R and we assume that $|G|$ is invertible in R so that $e = |G|^{-1}\Sigma_{x\in G}\bar{x}$ exists in the skew group ring RG. As we noted above, we may identify R^G with $eRGe \subset RG$. Hence, in view of the above remarks, we immediately deduce from Lemma 4.5 and from Theorem 4.4 that

$$\operatorname{rank} R^G \leqq \operatorname{rank} RG = \operatorname{rank} R.$$

Note that e is an idempotent of RG with $\operatorname{tr} e = |G|^{-1}$, certainly a unit in R. Thus if rank $R \geqq n$, for some integer n, then Lemma 4.3 implies that there exists a chain

$$P_1 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n$$

of prime (or primitive) ideals of RG with $e \notin P_n$. Hence, by Lemma 4.5, with $S = RG$ and $f = e$, we see that

$$P_0^{\varphi} \subsetneqq P_1^{\varphi} \subsetneqq \cdots \subsetneqq P_n^{\varphi}$$

is a chain of prime (or primitive) ideals of $eRGe \simeq R^G$. We deduce from this that rank $R \leqq$ rank R^G and we have equality throughout. The theorem is proved.

§5. Appendix: the infinite cyclic group

The goal of this final section is to obtain results analogous to Theorems 1.2 and 2.5 in case $G = Z$ is infinite cyclic. Obviously at this point we must allow G to be an infinite group. As we will see, the key properties of Z needed to make our arguments work are:

(i) Z is abelian.

(ii) Any nonidentity subgroup of Z has finite index.

These properties of course characterize Z among the infinite groups. The techniques used here are quite similar to those of Section 2. Therefore to avoid unnecessary repetition, we will mainly concentrate on the differences and merely sketch the proofs when similarities occur.

Let $R * G$ be given with G arbitrary and with R a G-prime ring. We let $\mathcal{F}$ denote the set of all nonzero G-invariant ideals of R. It is clear that $\mathcal{F}$ is closed under sums as well as finite products and intersections. Furthermore, each ideal in $\mathcal{F}$ has zero right and left annihilators in R. Using this collection of ideals, we can now define a ring of quotients $S = Q_G(R)$ quite analogous to $Q_0(R)$. Indeed S consists of all equivalence classes of left R-module homomorphisms $f : {}_RA \to {}_RR$ but here we insist that $A \in \mathcal{F}$. In view of the above properties of $\mathcal{F}$, it is clear that, with the natural addition and multiplication, S will be a ring extension of R.

LEMMA 5.1. *Let $S = Q_G(R)$ be as above.*

(i) *If $s \in S$ and $As = 0$ for some $A \in \mathcal{F}$, then $s = 0$.*

(ii) *If $s_1, s_2, \cdots, s_n \in S$ then there exists $A \in \mathcal{F}$ with $As_1, As_2, \cdots, As_n \subset R$.*

(iii) *Let σ be an automorphism of R such that $\mathcal{F}^\sigma = \mathcal{F}$. Then σ extends to a unique automorphism of S. In particular, each ${}^{\bar{x}}$ with $x \in G$ extends to a unique automorphism of S.*

(iv) *Let σ be an automorphism of R and let $A, B \in \mathcal{F}$. Suppose that $f : A \to B$ is an additive bijection which satisfies*

$$(rat)f = r(af)t^\sigma$$

for all $r, t \in R$ and $a \in A$. Then $s = \hat{f}$ is a unit in S, conjugation by s induces the automorphism σ on R and $af = as$ for all $a \in A$.

PROOF. Parts (i), (ii) and (iv) follow as in Lemma 2.1. Part (iii) follows as in [8, lemma 2.1 (iv)]. One needs $\mathcal{F} = \mathcal{F}^\sigma$ here in order to define $f^\sigma : {}_RA^\sigma \to {}_RR$. Since each ideal in $\mathcal{F}$ is ${}^{\bar{x}}$-invariant for all $x \in G$, it is now clear that each automorphism ${}^{\bar{x}}$ is uniquely extendable to S.

Using the observation of (iii) above we extend each ^{x} uniquely to an automorphism of S which we denote by the same symbol. We now define G_{inn} to be the set of all $x \in G$ such that the automorphism ^{x} is induced by conjugation by a unit of S. It follows easily that G_{inn} is a normal subgroup of G. Furthermore, if C denotes the center of S, then we have

LEMMA 5.2. *There exists a unique crossed product $S * G$ extending $R * G$. Moreover, if E denotes the centralizer of S in $S * G_{\text{inn}}$, then $S * G_{\text{inn}} = S \otimes_C E$ and $E = C^t[G_{\text{inn}}]$, some twisted group ring over the commutative ring C.*

PROOF. This follows as in [8, lemma 2.3]. Note that here we assume, by definition, that $E \subset S * G_{\text{inn}}$.

We remark that the ring S given here and hence also the group G_{inn} are different from the analogous objects defined in Section 2. Nevertheless we see that S enjoys properties quite similar to the ring $Q_0(R)$. The real difference occurs when we consider the center C. It is certainly not true that C must be a field. Indeed, if R is a G-simple ring, then $R = S$ and C is just the center of R. On the other hand, one can hope, as in [1, theorem 5], that C is a von Neumann regular ring or at least a ring with similar properties. We show this in the next lemma.

Observe that $\mathfrak{G}$ acts on R and on S and that U, the group of units in R, centralizes C. Thus $G \simeq \mathfrak{G}/U$ acts on C. Whenever we have a permutation action of G on any set Ω, an element $\omega \in \Omega$ is said to be G-orbital if and only if ω has only finitely many distinct G-translates. In particular, we can speak of G-orbital ideals of R and G-orbital elements of C.

LEMMA 5.3. *Let C denote the center of $S = Q_G(R)$.*

(i) *C is the centralizer of R in S and if $f : {}_RA \to {}_RR$ represents an element of S, then $\hat{f} \in C$ if and only if f is an R-bimodule homomorphism.*

(ii) *If N is a G-orbital nilpotent ideal of R, then $N = 0$.*

(iii) *If D is a G-orbital ideal of R, then there exist G-orbital ideals A, B of R such that $A \subset D$, $B \cap D = 0$ and $A \oplus B \in \mathcal{F}$.*

(iv) *Let $c \in C$ be a G-orbital element. Then there exists a G-orbital element $c' \in C$ such that $cc'c = c$.*

(v) *If $[G : G_{\text{inn}}] < \infty$, then C is a finite direct sum of fields and these fields are permuted transitively by G.*

PROOF. (i) Let $f : {}_RA \to {}_RR$ represent an element of S. If $\hat{f}$ centralizes R, then for all $r \in R$, since $r_\rho f$ and fr_ρ are both defined on A, we must have $r_\rho f = fr_\rho$ on A. In other words, for all $a \in A$

$$(ar)f = ar_\rho f = afr_\rho = (af)r$$

and f is an R-bimodule homomorphism.

Now suppose f is an R-bimodule homomorphism and let $g : {}_RB \to {}_RR$ represent an element of S. Then fg and gf are both defined on $D = AB \cap BA$ and for all $d \in D$ and $a \in A$ we have, since f is a bimodule homomorphism,

$$(ad)fg = ((af)d)g = (af)(dg)$$

and

$$(ad)gf = (a(dg))f = (af)(dg).$$

Thus fg and gf agree on $AD \in \mathcal{F}$ and we conclude that $\hat{f}$ is central.

(ii) If N is a G-orbital nilpotent ideal of R, then $\Sigma_{x \in G} N^{\bar{x}}$ is a G-invariant ideal of R which is also nilpotent since the sum is finite. But R is G-prime, so $\Sigma_{x \in G} N^{\bar{x}} = 0$ and $N = 0$.

(iii) Let D be G-orbital. If $D = 0$, take $A = 0$ and $B = R$. On the other hand, if $\bigcap_{x \in G} D^{\bar{x}} \neq 0$, take $A = \bigcap_{x \in G} D^{\bar{x}}$ and $B = 0$. Thus we may assume that $D \neq 0$ but that $\bigcap_{x \in G} D^{\bar{x}} = 0$.

Observe that $\bigcap_{x \in G} D^{\bar{x}}$ is actually a finite intersection since D is G-orbital. Hence since $D \neq 0$ there exists a subset $X \subset G$ maximal with respect to the property that $V = \bigcap_{x \in X} D^{\bar{x}} \neq 0$. Again, since the latter intersection is also finite, we see that V is G-orbital with distinct conjugates $V = V_1, V_2, \cdots, V_k$.

Note that the maximality of X implies that $V_i V_j \subset V_i \cap V_j = 0$ for $i \neq j$. Hence for each i, $V_i \cap (\Sigma_{j \neq i} V_j)$ is a G-orbital ideal of square zero. We therefore conclude from (ii) above that each such intersection is zero and thus $\Sigma_1^n V_i$ is a direct sum.

Let $A = \Sigma' V_i$ be the partial sum of those V_i with $V_i \subset D$ and let $B = \Sigma'' V_i$ be the sum of the remaining terms. Then clearly $A \oplus B \in \mathcal{F}$ and $A \subset D$. Moreover, again by the maximality of X, $DV_i \subset D \cap V_i = 0$ for all V_i not contained in D. Hence $D \cap B$ is a G-orbital ideal of square zero, so $D \cap B = 0$ and this part is proved.

(iv) Let $c \in C$ be represented by the R-module homomorphism $f : {}_RM \to {}_RR$ with $M \in \mathcal{F}$. Since f is an R-bimodule homomorphism, by (i) above, $D = \ker f$ is a two sided ideal of R contained in M. Note that $f^{\bar{x}}$ is defined on M since M is G-invaraint. Hence since $f^{\bar{x}}$ represents $c^{\bar{x}}$ and c is G-orbital, we conclude that D is a G-orbital ideal of R.

By (iii), there exist G-orbital ideals A, B_1 of R with $A \subset D$, $B_1 \cap D = 0$ and $A \oplus B_1 \in \mathcal{F}$. Note that $M \supset D \supset A$ so

$$(A \oplus B_1) \cap M = A \oplus (B_1 \cap M) \in \mathcal{F}$$

and we set $B = B_1 \cap M$. Of course, B is also G-orbital. Since $B \cap D = 0$ and $B \subset M$, the restricted map $f : {}_R B \to {}_R R$ is clearly one-to-one. Furthermore, since f is an R-bimodule homomorphism on $M \supset B$ and since both f and B are G-orbital, it follows immediately that Bf is also a G-orbital ideal of R.

We can now apply (iii) above to the ideal Bf and we conclude that there exist G-orbital ideals V, W of R with $V \subset Bf$, $W \cap Bf = 0$ and $V \oplus W \in \mathcal{F}$. Let $g : {}_R(V \oplus W) \to {}_R R$ be defined by $(v + w)g = vf^{-1}$ where f^{-1} denotes the well defined back map $f^{-1} : Bf \to B$. Then certainly g is a G-orbital R-bimodule homomorphism and since $V \oplus W \in \mathcal{F}$ we see that $c' = \hat{g}$ is a G-orbital element of C. Finally let $F = (V \oplus W)(A \oplus B) \in \mathcal{F}$. Then $Ff \subset V \oplus W$ and since $Ff \subset Bf$ we have

$$Ff \subset (V \oplus W) \cap Bf = V.$$

With this, it is now a simple matter to compute fgf on F. Indeed if $a + b \in F \subset A \oplus B$, with $a \in A$, $b \in B$, then

$$\begin{aligned}(a+b)fgf &= (bf)gf \\ &= (bf)f^{-1} \cdot f \\ &= bf \\ &= (a+b)f.\end{aligned}$$

Thus fgf and f agree on $F \in \mathcal{F}$ and we conclude that

$$cc'c = \hat{f}\hat{g}\hat{f} = \hat{f} = c.$$

(v) We note that G_{inn} certainly acts trivially on C. Thus if $[G : G_{\mathrm{inn}}] < \infty$, then every element of C is G-orbital and we conclude from (iv) that C is a commutative von Neumann regular ring. Now it is clear from Lemma 5.1 (i), (ii) that every nonzero G-invariant ideal of S intersects R in a nonzero G-invariant ideal. Thus S is also G-prime and this implies easily that C is G-prime. In fact, more can be said. Suppose I is a nonzero G-invariant ideal of C. Since G acts as a finite group on C it is clear that I contains a nonzero finitely generated G-invariant ideal J. But C is a commutative regular ring, so J must be generated by an idempotent. Moreover C is G-prime so the annihilator of J in C must be zero and hence we see that $J = C$ and $I = C$. In other words, we conclude that C

is G-simple. Finally let M be a maximal ideal of C. Then the finite intersection $\bigcap_{x\in G} M^{\bar{x}}$ is a proper G-invariant ideal of C which must therefore be zero. This implies immediately that C is a finite direct sum of fields and then, since C is G-simple, that G must permute these summands transitively.

We can now begin our work on $R * G$.

PROPOSITION 5.4. *Let $R * G$ be a crossed product with G abelian and with R a G-prime ring. If $G_{\text{inn}} = \langle 1 \rangle$, then $R * G$ is a prime ring and every nonzero ideal of $R * G$ has nonzero intersection with R.*

PROOF. Let I be a nonzero ideal of $R * G$ and let γ be a nonzero element of I of minimal support size and with $1 \in \operatorname{Supp} \gamma$. Then it is clear that $T = \operatorname{Supp} \gamma$ satisfies the hypothesis of Lemma 1.5, a result which holds equally well for infinite groups. We apply this lemma and use its notation. But observe that since G is abelian, T is a central subset of G and hence each of the ideals A and B_i obtained is nonzero and G-invariant. Thus the functions $f_i : A \to B_i$ satisfy the hypothesis of Lemma 5.1 (iv) with $\sigma = \bar{x}_i^{-1}$ and we conclude that $x_i \in G_{\text{inn}}$ for all i. But $G_{\text{inn}} = \langle 1 \rangle$ so $T = \{1\}$ and $0 \neq I \cap R * T = I \cap R$.

Finally if I and J are nonzero ideals of $R * G$, then $I \cap R$ and $J \cap R$ are nonzero G-invariant ideals of R so the G-primeness of R yields $0 \neq (I \cap R)(J \cap R) \subset IJ$. Thus $R * G$ is prime.

We now further restrict our attention to the case where $G = Z = \langle x \rangle$ is infinite cyclic. If $Z_{\text{inn}} = \langle 1 \rangle$, then the preceding result shows that 0 is the unique prime ideal of $R * Z$ disjoint from $R \setminus \{0\}$ and every ideal $I \supsetneqq 0$ meets R nontrivially. Thus we need only consider the possibility that $Z_{\text{inn}} \neq \langle 1 \rangle$ and of course this implies that $[Z : Z_{\text{inn}}] < \infty$. We first consider the structure of $E = C^t[Z_{\text{inn}}]$.

Observe that if $[Z : Z_{\text{inn}}] < \infty$, then by Lemma 5.3 (v), $C = e_1C \oplus e_2C \oplus \cdots \oplus e_mC$ is a finite direct sum of fields $F_i = e_iC$ and that these summands are permuted transitively by Z. Set $e = e_1$, $F = F_1$ and let H be the stabilizer of e in Z. Since Z is abelian, H of course stabilizes all the idempotents e_i. Furthermore we have $Z \supset H \supset Z_{\text{inn}}$ and since $[Z : H] = m$ we know that $\{1, x, \cdots, x^{m-1}\}$ is a transversal for H in Z.

LEMMA 5.5. *Suppose that $Z_{\text{inn}} \neq \langle 1 \rangle$. Then with the above notation we have*

(i) *$E = e_1E \oplus e_2E \oplus \cdots \oplus e_mE$ and $e_iE \simeq eE = F[Z_{\text{inn}}]$, where the latter is the ordinary group algebra of Z_{inn} over F.*

(ii) *The maps* $L \to eL$ *and* $J \to \Sigma_{i=0}^{m-1} J^{\bar{x}^i}$ *yield a one-to-one correspondence between the* Z*-invariant ideals* L *of* E *and the* H*-invariant ideals* J *of* $F[Z_{\rm inn}]$. *Furthermore, in this way the* Z*-prime ideals of* E *correspond to the* H*-prime ideals of* $F[Z_{\rm inn}]$.

PROOF. Since Z permutes the idempotents e_i transitively it follows from Lemma 5.2 that $e_iE \simeq eE = eC^t[Z_{\rm inn}] = F^t[Z_{\rm inn}]$. Moreover if $Z_{\rm inn} = \langle z \rangle$ and if $\tilde{z} \in F^t[Z_{\rm inn}]$ corresponds to z in this twisted group algebra, then $F^t[Z_{\rm inn}] = F[\tilde{z}, 1/\tilde{z}] \simeq F[Z_{\rm inn}]$. This proves (i) and part (ii) is obvious since $e_i = e^{\bar{x}^{i-1}}$ and since any ideal L of E satisfies $L = \Sigma_i e_i L$.

We now define the appropriate maps ^{u} and ^{d} as follows.

DEFINITION. (i) If L is a Z-invariant ideal of E, then we set

$$L^u = L(S * Z) \cap R * Z$$

so that L^u is an ideal of $R * Z$.

(ii) For any ideal I of $R * Z$ we set

$$I^d = \{\gamma \in E \mid A\gamma \subset I \text{ for some } A \in \mathcal{F}\}.$$

Then I^d is certainly a Z-invariant ideal of E.

We remark that any crossed product of Z is clearly just a skew group ring once we make an obvious change of basis. Thus there is really no additional generality gained in dealing with crossed products. The following is the main result of this section.

THEOREM 5.6. *Let* $R * Z$ *be a crossed product of the infinite cyclic group* Z *over the* Z*-prime ring* R *and let* $E = C^t[Z_{\rm inn}]$ *be the centralizer of* $S = Q_Z(R)$ *in* $S * Z_{\rm inn}$. *If* $Z_{\rm inn} \neq \langle 1 \rangle$, *then the maps* ^{d} *and* ^{u} *yield a one-to-one correspondence between the prime ideals* P *of* $R * Z$ *with* $P \cap R = 0$ *and the* Z*-prime ideals of* E. *More precisely*

(i) *If* P *is a prime ideal of* $R * Z$ *with* $P \cap R = 0$, *then* P^d *is a* Z*-prime ideal of* E *and* $P = P^{du}$.

(ii) *If* L *is a* Z*-prime ideal of* E, *then* L^u *is a prime ideal of* $R * Z$ *with* $L^u \cap R = 0$ *and* $L = L^{ud}$.

PROOF. Since $Z_{\rm inn} \neq \langle 1 \rangle$, Lemma 5.5 and its notation applies. We now proceed as in Section 2.

Let L, L_1 and L_2 be Z-invariant ideals of E. Observe that eL has an F-complement K in eE and thus L has a C-complement $L' = \Sigma_0^{m-1} K^{\bar{z}^i}$ in E. With this, it follows as in Lemmas 2.2 and 2.3 that L^u is R-cancelable and $L_1^u \cdot L_2^u \subset (L_1L_2)^u$. Of course the inclusion $L \subset L^{ud}$ is trivially true. Furthermore if $L \neq E$, then $eL \neq eE$ and we can assume that $e \in K$. Thus $1 = \Sigma_0^{m-1} e^{\bar{z}^i} \in L'$ and we conclude that if $L \neq E$, then $L^u \cap R = 0$.

Now let I be an ideal of $R * Z$. The crux of the argument is to show that $I \subset I^{du}$ and this proceeds as in Lemma 2.4 (ii). We know that I^{du} is R-cancelable. Furthermore, when we consider the minimal subset T in that proof, we observe that here T is a central subset of the abelian group Z. Therefore all the ideals A and B_i are necessarily Z-invariant and Lemma 5.1 (iv) applies to yield the result.

Now let P be a prime ideal of $R * Z$ with $P \cap R = 0$. Since $\mathcal{F}$ consists of Z-invariant ideals of R, it follows precisely as in the proof of Theorem 2.5 (i) that P^d is a Z-invariant prime ideal of E and that $P = P^{du}$.

Finally we study the Z-prime ideals L of E and here we consider separately the cases $L \neq 0$ and $L = 0$. Note that $L = 0$ is indeed a Z-prime ideal of E by Lemma 5.5. Suppose $L \neq 0$. Then it follows immediately from Lemma 5.5 that E/L is an Artinian ring and hence that L is in fact a Z-maximal ideal of E. With this observation, the proof of Theorem 2.5 (ii) goes over to show that L^u is a prime ideal of $R * Z$ with $L^u \cap R = 0$, that $L = L^{ud}$ and furthermore that if $I \supsetneqq L^u$ then $I \cap R \neq 0$. On the other hand, if $L = 0$, then certainly $L^u = 0$ and $L = L^{ud}$. It remains to show that $0 = L^u$ is a prime ideal of $R * Z$. This is of course well known, but a simple argument is as follows. Suppose $I_1 I_2 = 0$. Then it is easy to see that $I_1^d I_2^d = 0$ and hence $I_j^d = 0$ for some j. But $I_j \subset I_j^{du} = 0$ so $I_j = 0$ and the result follows.

In addition, we have

LEMMA 5.7. *Let P be a nonzero prime ideal of $R * Z$ where R is Z-prime. If $I \supsetneqq P$, then $I \cap R \neq 0$.*

PROOF. We may clearly assume that $P \cap R = 0$ and therefore, by Proposition 5.4, that $Z_{\text{inn}} \neq \langle 1 \rangle$. Now we know that $P = L^u$ with L a nonzero Z-prime ideal of E and thus, as we showed in the course of the preceding proof, $I \supsetneqq L^u = P$ implies that $I \cap R \neq 0$.

We can now obtain our infinite analog of Theorem 1.2. As is well known, in this situation, ordinary incomparability does not hold.

THEOREM 5.8. *Let $R * G$ be a crossed product where G is a group with a cyclic subgroup of finite index. Suppose $P_1 \subsetneqq P_2 \subsetneqq I$ are three ideals of $R * G$ with P_1 and P_2 prime. Then*

$$P_1 \cap R \neq I \cap R.$$

PROOF. If $I \cap R$ is not a G-prime ideal of R, then certainly $P_1 \cap R \neq I \cap R$. Thus we may assume that $I \cap R$ is G-prime. Now let P_3 be an ideal of $R * G$ maximal with respect to the property that $P_3 \supset I$ and $P_3 \cap R = I \cap R$. The assumption on $I \cap R$ implies immediately that P_3 is prime. Since $P_3 \supset I \supsetneqq P_2$ and $P_3 \cap R = I \cap R$, we can now assume without loss of generality that $I = P_3$ is a prime ideal.

In view of Theorem 1.2, we may clearly also assume that G is infinite. Then it follows easily that G has a normal infinite cyclic subgroup Z of finite index. Observe that $R * G = (R * Z) * \tilde{G}$, where the latter is a suitable crossed product of the finite group $\tilde{G} = G/Z$ over the ring $R * Z$. Therefore, by Theorem 1.2, we know at least that

$$P_1 \cap R * Z \subsetneqq P_2 \cap R * Z \subsetneqq P_3 \cap R * Z.$$

Furthermore, each of these intersections is surely a $\tilde{G}$-prime ideal of $R * Z$.

Now for each $\tilde{x} \in \tilde{G} = G/Z$ let us choose a fixed coset representative $x \in G$. With this notation, since $(R * Z)/(P_i \cap R * Z)$ is a $\tilde{G}$-prime ring, it follows from Lemma 3.1 (i) that there exists a prime ideal Q_i of $R * Z$ with

$$P_i \cap R * Z = \bigcap_{\tilde{x} \in \tilde{G}} Q_i^{\tilde{x}}.$$

Observe that $Q_3 \supset \bigcap Q_2^{\tilde{x}}$ and hence $Q_3 \supset Q_2^{\tilde{y}}$ for some $\tilde{y} \in \tilde{G}$. Thus by relabeling, if necessary, we can assume that $Q_3 \supset Q_2$. Furthermore, we cannot have $Q_3 = Q_2$ here since this would yield

$$P_3 \cap R * Z = \bigcap Q_3^{\tilde{x}} = \bigcap Q_2^{\tilde{x}} = P_2 \cap R * Z,$$

a contradiction. Thus $Q_3 \supsetneqq Q_2$ and similarly, by relabeling if necessary, we can assume that $Q_2 \supsetneqq Q_1$.

Let $A = Q_1 \cap R$. Then R/A is a Z-prime ring and we have $Q_3/A * Z \supsetneqq Q_2/A * Z \neq 0$ in the crossed product $(R/A) * Z = (R * Z)/(A * Z)$. Thus we conclude from Lemma 5.7 that $(Q_3/A * Z) \cap (R/A) \neq 0$ or equivalently that $Q_3 \cap R \supsetneqq A = Q_1 \cap R$. Suppose that $Q_3 \cap R \supset (Q_3 \cap R)^{\tilde{x}}$ for some $\tilde{x} \in \tilde{G}$. Then by applying ${}^{\tilde{x}^k}$ to the above inclusion, we deduce that $(Q_3 \cap R)^{\tilde{x}^k} \supset (Q_3 \cap R)^{\tilde{x}^{k+1}}$ for all $k \geqq 0$. But $\tilde{G}$ is finite, so $\tilde{x}^m = 1$ for some $m \geqq 1$, and hence

$(Q_3 \cap R)^{\bar{x}^m} = Q_3 \cap R$. We conclude from this that we have equality throughout and in particular that $Q_3 \cap R = (Q_3 \cap R)^{\bar{x}}$. Thus since $Q_3 \cap R \supsetneqq Q_1 \cap R$, we see that $Q_1 \cap R \not\supseteq (Q_3 \cap R)^{\bar{x}}$ for all $\bar{x} \in \bar{G}$. But $Q_1 \cap R$ is a Z-prime ideal of R and each $(Q_3 \cap R)^{\bar{x}}$ is Z-invariant. Thus we see that $Q_1 \cap R \not\supseteq \bigcap_{\bar{x} \in \bar{G}} (Q_3 \cap R)^{\bar{x}}$.

Finally observe that

$$\begin{aligned} P_3 \cap R &= (P_3 \cap R * Z) \cap R \\ &= \Big(\bigcap_{\bar{x} \in \bar{G}} Q_3^{\bar{x}} \Big) \cap R \\ &= \bigcap_{\bar{x} \in \bar{G}} (Q_3 \cap R)^{\bar{x}} \end{aligned}$$

and similarly

$$P_1 \cap R = \bigcap_{\bar{x} \in \bar{G}} (Q_1 \cap R)^{\bar{x}} \subseteq Q_1 \cap R.$$

Thus the result of the preceding paragraph shows that $P_1 \cap R \not\supseteq P_3 \cap R$. Hence $P_1 \cap R \neq P_3 \cap R$ and the theorem is proved.

This completes our work on crossed products. We close this paper with a brief comment on skew polynomial rings. Let $R[x;\sigma]$ be a skew polynomial ring with σ an automorphism of R. Then we can certainly embed this ring, in a natural manner, in the skew group algebra $R\langle x \rangle$. Furthermore, if P is a prime ideal of $R[x;\sigma]$ with $x \notin P$, then it is fairly easy to see that $\bar{P} = R\langle x \rangle \cdot P$ is a two sided ideal of $R\langle x \rangle$ which is prime and satisfies $\bar{P} \cap R[x;\sigma] = P$. Since the work of this section yields information on the structure of $\bar{P}$, it now also yields information on the structure of P. In particular, we can use this idea to obtain an incomparability result for skew polynomial rings analogous to Theorem 5.8. We remark however that this incomparability theorem is due to Bergman, who offers a direct computational proof which is, at present, unpublished.

Added in Proof. In a recent paper on group rings of polycyclic-by-finite groups, we have observed that the ν map on the appropriate primes can be alternately characterized as the induced ideal map. Furthermore, we have since realized that a considerable simplification in the work of Section 3 can be achieved by dealing directly with the latter map. Indeed, the unpleasant computations of Lemmas 2.8, 3.5 and 3.8 can all be eliminated in this manner. We plan to exhibit such an alternate proof in an addendum to this paper.

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